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Supplementary material for 'Exact simulation of max-stable processes'

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1. PROOF OF PROPOSITION 9

Proof of Proposition 9. In order to analyze the complexity of Algorithm 1, we consider each step of the algorithm separately. In the nth step, i.e. for sampling the process perfectly at site x_n, we simulate Poisson points ζ and stochastic processes Y, until one of the following two conditions is satisfied:

(a) $\zeta < Z_{n-1}(x_n)$. This condition is checked directly after the simulation of ζ and, in this case, no stochastic process Y needs to be simulated.
(b) $\zeta > Z_{n-1}(x_n)$ and $\zeta Y(x_i) \leq Z(x_i)$ for all $1 \leq i < n-1$. In this case, Z is updated and ζY is an extremal function as it contributes to Z at site x_n (and possibly also at some of the sites $x_{n+1}, \dots, x_N$).

Thus, any stochastic process that is simulated is either rejected, i.e. it is not considered as contribution to Z as it does not respect all the values $Z(x_1), \dots, Z(x_{n-1})$, or it leads to an extremal function. Denoting by $\Phi^{(n)} = \{(\xi_i^{(n)}, \psi_i^{(n)}), i \geq 1\}$ a Poisson point process on $(0, \infty) \times \mathcal{C}$ with intensity measure $\xi^{-2}\mathrm{d}\xi\, P_{x_n}(\mathrm{d}\psi)$, the random number $C_1(N)$ of processes simulated in Algorithm 1 satisfies

$$C_1(N) = |\Phi^+_{\{x_1,\dots,x_N\}}| + \sum_{n=2}^{N} \left| \left\{ i \geq 1 : \ \xi_i^{(n)} > Z(x_n),\ \xi_i^{(n)} > \min_{j=1}^{n-1} \frac{Z(x_j)}{\psi_i^{(n)}(x_j)} \right\} \right|. \qquad (1)$$

In this formula, the term $|\Phi^+_{\{x_1,\dots,x_N\}}|$ is the number of extremal functions that need to be simulated, and the term with index n in the sum is the number of functions that are simulated at the nth step but rejected since $\xi_i^{(n)} \psi_i^{(n)}(x_j) > Z(x_j)$ for some $j \leq n-1$. For the computation of the expectation of the second term, conditionally on $\Phi^+_{\{x_1,\dots,x_{n-1}\}}$, i.e. for fixed $Z(x_j)$, $1 \leq j \leq n-1$, the two sets

$$\Phi_1^{(n)} = \{(\xi_i^{(n)}, \psi_i^{(n)}) : \xi_i^{(n)}\psi_i^{(n)}(x_j) > Z(x_j) \text{ for some } j = 1, \dots, n-1\},$$
$$\Phi_2^{(n)} = \{(\xi_i^{(n)}, \psi_i^{(n)}) : \xi_i^{(n)}\psi_i^{(n)}(x_j) \le Z(x_j) \text{ for all } j = 1, \dots, n-1\}$$

are restrictions of the Poisson point process $\Phi^{(n)}$ to disjoint sets and, thus, are independent Poisson point processes with intensities $\xi^{-2}1_{\left\{\xi > \min_{j=1}^{n-1}(Z(x_j)/\psi(x_j))\right\}}\mathrm{d}\xi\, P_{x_n}(\mathrm{d}\psi)$ and $\xi^{-2}1_{\left\{\xi < \min_{j=1}^{n-1}(Z(x_j)/\psi(x_j))\right\}}\mathrm{d}\xi\, P_{x_n}(\mathrm{d}\psi)$, respectively. Conditioning further on $\Phi_2^{(n)}$, $Z(x_n)$ is also fixed and we obtain

$$\begin{aligned}
&E\left[\left|\left\{(\xi_i^{(n)}, \psi_i^{(n)}) : \xi_i^{(n)} > Z(x_n),\, \xi_i^{(n)} > \min_{j=1}^{n-1} \frac{Z(x_j)}{\psi_i^{(n)}(x_j)}\right\}\right|\right] \\
&= E\left(E\left[\left|\left\{(\xi,\psi) \in \Phi_1^{(n)} : \xi > Z(x_n)\right\}\right| \,\middle|\, \Phi^+_{\{x_1,\dots,x_{n-1}\}},\, \Phi_2^{(n)}\right]\right) \\
&= E\left[\int\int \xi^{-2} 1_{\{\xi > Z(x_n)\}} 1_{\left\{\xi > \min_{j=1}^{n-1} \frac{Z(x_j)}{\psi(x_j)}\right\}} \mathrm{d}\xi\, P_{x_n}(\mathrm{d}\psi)\right] \\
&= E\left[\min\left\{\frac{1}{Z(x_n)}, \max_{j=1}^{n-1} \frac{Y_n(x_j)}{Z(x_j)}\right\}\right]
\end{aligned}$$

where $Y_n \sim P_{x_n}$ and Z are independent. The relation $\min\{a,b\} = a + b - \max\{a,b\}$, $a, b \in \mathbb{R}$, and the fact that $Y_n(x_n) = 1$ almost surely yield

$$\begin{aligned}
&E\left[\left|\left\{(\xi_i^{(n)}, \psi_i^{(n)}) : \xi_i^{(n)} > Z(x_n),\, \xi_i^{(n)} > \min_{j=1}^{n-1} \frac{Z(x_j)}{\psi_i^{(n)}(x_j)}\right\}\right|\right] \\
&= E\left\{\frac{1}{Z(x_n)}\right\} + E\left\{\max_{j=1}^{n-1} \frac{Y_n(x_j)}{Z(x_j)}\right\} - E\left\{\max_{j=1}^{n} \frac{Y_n(x_j)}{Z(x_j)}\right\} \\
&= 1 + E\left(|\Phi^+_{\{x_1,\dots,x_{n-1}\}}|\right) - E\left(|\Phi^+_{\{x_1,\dots,x_n\}}|\right),
\end{aligned}$$

as $E\left(|\Phi^+_{\{x_1,\dots,x_n\}}|\right) = E\left\{\max_{j=1}^n Y_n(x_j)/Z(x_j)\right\}$ by Lemma 4.7 in the 2013 technical report by M. Oesting, M. Schlather and C. Zhou (arXiv:1310.1813v1). Thus, by (1), we obtain

$$\begin{aligned}
E\{C_1(N)\} &= E\left(|\Phi^+_{\{x_1,\dots,x_N\}}|\right) + \sum_{n=2}^N \left\{1 + E\left(|\Phi^+_{\{x_1,\dots,x_{n-1}\}}|\right) - E\left(|\Phi^+_{\{x_1,\dots,x_n\}}|\right)\right\} \\
&= N - 1 + E\left(|\Phi^+_{\{x_1\}}|\right) = N.
\end{aligned}$$

Moreover, by (2), we have that $E\{Z(x_i)^{-1}\} = 1$ for $i = 1, \dots, N$, and, thus,

$$E\left\{\max_{i=1}^N Z(x_i)^{-1}\right\} \ge 1,$$

with equality if only if $Z(x_1) = \cdots = Z(x_N)$ holds almost surely. □

2. Proofs for Section 4

2·1. *Moving maximum process*

Proof of Proposition 3. In the case of the moving maximum process (12), the measure ν associated with the representation (1) is

$$\nu(A) = \int_{\mathcal{X}} 1_{\{h(\cdot - \chi)\in A\}} \lambda(\mathrm{d}\chi), \quad A \subset \mathcal{C} \text{ Borel}.$$

We deduce from Proposition 1,

$$\begin{aligned} P_{x_0}(A) &= \int_{\mathcal{C}} 1_{\{f/f(x_0)\in A\}} f(x_0)\nu(\mathrm{d}f) = \int_{\mathcal{X}} 1_{\{h(\cdot-\chi)/h(x_0-\chi)\in A\}} h(x_0-\chi)\lambda(\mathrm{d}\chi) \\ &= \int_{\mathcal{X}} 1_{\{h(\cdot+u-x_0)/h(u)\in A\}} h(u)\lambda(\mathrm{d}u) \end{aligned}$$

where the last line follows from the simple change of variable $x_0 - \chi = u$. This proves the result since $h(u)\lambda(\mathrm{d}u)$ is a density function on $\mathcal{X}$. □

2·2. *Brown–Resnick process*

Our proof of Proposition 4 relies on the following lemma on exponential changes of measures for Gaussian processes. Note that the distribution of P_{x_0} is strongly connected to the notion of conditional intensity introduced in Dombry et al. (2013) and that the formula are similar.

LEMMA 1. *The distribution of the random process $(W(x))_{x\in\mathcal{X}}$ under the transformed probability measure $\widehat{\mathrm{pr}} = e^{W(x_0)-\sigma^2(x_0)/2}\mathrm{dpr}$ is equal to the distribution of the Gaussian random process*

$$W(x) + K(x_0, x), \quad x \in \mathcal{X},$$

where $K(x, y)$ denotes the covariance between $W(x)$ and $W(y)$.

Proof of Lemma 1. We need to consider finite dimensional distributions only and we compute for some $x_1, \ldots, x_k \in \mathcal{X}$ the Laplace transform of $(W(x_i))_{1\le i\le k}$ under the transformed probability measure $\widehat{\mathrm{pr}}$. For all $\theta = (\theta_1, \ldots, \theta_k) \in \mathbb{R}^k$, we have

$$\begin{aligned} \mathcal{L}(\theta_1, \ldots, \theta_k) &= \widehat{E}\left\{e^{\sum_{i=1}^k \theta_i W(x_i)}\right\} = E\left\{e^{W(x_0)-\sigma^2(x_0)/2} e^{\sum_{i=1}^k \theta_i W(x_i)}\right\} \\ &= \exp\left(\frac{1}{2}\tilde{\theta}^{\mathrm{T}}\tilde{\Sigma}\tilde{\theta} - \frac{1}{2}\sigma^2(x_0)\right), \end{aligned} \tag{2}$$

with $\tilde{\theta} = (1, \theta) \in \mathbb{R}^{k+1}$ and $\tilde{\Sigma} = (K(x_i, x_j))_{0\le i,j\le k}$ the covariance matrix. We introduce the block decomposition

$$\tilde{\Sigma} = \begin{pmatrix} \sigma^2(x_0) & \Sigma_{0,k} \\ \Sigma_{k,0} & \Sigma \end{pmatrix}$$

with $\Sigma = (K(x_i, x_j))_{1\le i,j\le k}$ and $\Sigma_{k,0} = \Sigma_{0,k}^{\mathrm{T}} = (K(x_0, x_i))_{1\le i\le k}$. The quadratic form in Equation (2) can be rewritten as

$$\frac{1}{2}\tilde{\theta}^{\mathrm{T}}\tilde{\Sigma}\tilde{\theta} - \frac{1}{2}\sigma^2(x_0) = \frac{1}{2}\left\{\sigma^2(x_0) + \theta^{\mathrm{T}}\Sigma\theta + 2\theta^{\mathrm{T}}\Sigma_{k,0}\right\} - \frac{1}{2}\sigma^2(x_0) = \theta^{\mathrm{T}}\Sigma_{k,0} + \frac{1}{2}\theta^{\mathrm{T}}\Sigma\theta.$$

We recognize the Laplace transform of a Gaussian random vector with mean $\Sigma_{k,0}$ and covariance matrix Σ whence the Lemma follows. □

Proof of Proposition 4. Equations (5) and (13) together with Lemma 1 yield, for all Borel set $A \subset \mathcal{C}$,

$$\begin{aligned} P_{x_0}(A) &= \int_{\mathcal{C}} 1_{\{f/f(x)\in A\}} f(x)\nu(\mathrm{d}f) = E\left[e^{W(x_0)-\frac{1}{2}\sigma^2(x_0)} 1_{\{e^{W(\cdot)-\frac{1}{2}\sigma^2(\cdot)}/e^{W(x_0)-\frac{1}{2}\sigma^2(x_0)}\in A\}}\right] \\ &= \widehat{\mathrm{pr}}\left(\exp\left[W(\cdot) - W(x_0) - \frac{1}{2}\left\{\sigma^2(\cdot) - \sigma^2(x_0)\right\}\right] \in A\right) \\ &= \mathrm{pr}\left(\exp\left[W(\cdot) + K(x_0,\cdot) - W(x_0) - K(x_0,x_0) - \frac{1}{2}\left\{\sigma^2(\cdot) - \sigma^2(x_0)\right\}\right] \in A\right) \\ &= \mathrm{pr}\left(\exp\left[W(\cdot) - W(x_0) - \frac{1}{2}\left\{\sigma^2(\cdot) + \sigma^2(x_0) - 2K(x_0,\cdot)\right\}\right] \in A\right). \end{aligned}$$

Using the fact that for all $x \in \mathcal{X}$

$$\sigma^2(x) + \sigma^2(x_0) - 2K(x_0,x) = \mathrm{var}\{W(x) - W(x_0)\}$$

we deduce that P_{x_0} is equal to the distribution of the log-normal process

$$\widetilde{Y}(x) = \exp\left[W(x) - W(x_0) - \frac{1}{2}\mathrm{var}\left\{W(x) - W(x_0)\right\}\right], \quad x \in \mathcal{X}.$$

This proves Proposition 4. □

2·3. *Extremal-t process*

It is worth noting that the formula for P_{x_0} provided in Proposition 5 is similar to the formula for the conditional intensity of the extremal-t process that was computed in Ribatet (2013). In the sequel, we write shortly $z_+^\alpha = \max(0,z)^\alpha$ for all real numbers z.

LEMMA 2. *The distribution of the random process $(W(x)/W(x_0))_{x\in\mathcal{X}}$ under the transformed probability measure $\widehat{\mathrm{pr}} = c_\alpha W(x_0)_+^\alpha \mathrm{dpr}$ is equal to the distribution of a Student process with $\alpha+1$ degrees of freedom, location parameter μ_k and scale matrix $\widehat{\Sigma}_k$ given by*

$$\mu_k = \Sigma_{k,0}, \quad \widehat{\Sigma}_k = \frac{\Sigma_k - \Sigma_{k,0}\Sigma_{0,k}}{\alpha+1},$$

where $\Sigma_k = (K(x_i,x_j))_{1\le i,j\le k}$ and $\Sigma_{k,0} = \Sigma_{0,k}^{\mathrm{T}} = (K(x_0,x_i))_{1\le i\le k}$.

Proof of Lemma 2. We consider finite dimensional distributions only. Let $k \geq 1$ and $x_1,\ldots,x_k \in \mathcal{X}$. We first assume that the covariance matrix $\widetilde{\Sigma} = (K(x_i,x_j))_{0\le i,j\le k}$ is non singular so that $(W(x_i))_{0\le i\le k}$ has density

$$\tilde{g}(y) = (2\pi)^{-(k+1)/2}\det(\widetilde{\Sigma})^{-1/2}\exp\left(-\frac{1}{2}y^{\mathrm{T}}\widetilde{\Sigma}^{-1}y\right), \quad y = (y_i)_{0\le i\le k}.$$

Setting $z = (y_i/y_0)_{1\le i\le k}$, we have for all Borel sets $A_1,\ldots,A_k \subset \mathbb{R}$

$$\begin{aligned} \widehat{\mathrm{pr}}\left\{\frac{W(x_i)}{W(x_0)} \in A_i,\ i=1,\ldots,k\right\} &= \int_{\mathbb{R}^{k+1}} 1_{\{y_i/y_0\in A_i,\ i=1,\ldots,k\}} c_\alpha (y_0)_+^\alpha \tilde{g}(y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^k} 1_{\{z_i\in A_i,\ i=1,\ldots,k\}} \left\{\int_0^\infty c_\alpha (y_0)_+^\alpha \tilde{g}(y_0, y_0 z)\, y_0^k \mathrm{d}y_0\right\}\mathrm{d}z \end{aligned}$$

We deduce that under $\widehat{\mathrm{pr}}$, the random vector $(W(x_i)/W(x_0))_{1\leq i\leq k}$ has density

$$\begin{aligned}g(z) &= \int_0^\infty c_\alpha y_0^{k+\alpha}\tilde{g}(y_0, y_0 z)\,\mathrm{d}y_0 \\ &= c_\alpha (2\pi)^{-(k+1)/2}\det(\widetilde{\Sigma})^{-1/2}\int_0^\infty y_0^{k+\alpha}\exp\left(-\frac{\tilde{z}^{\mathrm{T}}\widetilde{\Sigma}^{-1}\tilde{z}}{2}y_0^2\right)\mathrm{d}y_0\end{aligned}$$

with $\tilde{z} = (1, z)$. Using the change of variable $u = \frac{\tilde{z}^{\mathrm{T}}\widetilde{\Sigma}^{-1}\tilde{z}}{2}y_0^2$, we get

$$\begin{aligned}\int_0^\infty y_0^{k+\alpha}\exp\left(-\frac{\tilde{z}^{\mathrm{T}}\widetilde{\Sigma}^{-1}\tilde{z}}{2}y_0^2\right)\mathrm{d}y_0 &= \frac{1}{2}\left(\frac{\tilde{z}^{\mathrm{T}}\widetilde{\Sigma}^{-1}\tilde{z}}{2}\right)^{-\frac{\alpha+k+1}{2}}\int_0^\infty u^{(k+\alpha-1)/2}\exp(-u)\,\mathrm{d}u \\ &= \frac{1}{2}\left(\frac{\tilde{z}^{\mathrm{T}}\widetilde{\Sigma}^{-1}\tilde{z}}{2}\right)^{-\frac{\alpha+k+1}{2}}\Gamma\left(\frac{k+\alpha+1}{2}\right)\end{aligned}$$

and we obtain after simplification

$$g(z) = \pi^{-k/2}\frac{\Gamma\left(\frac{k+\alpha+1}{2}\right)}{\Gamma\left(\frac{\alpha+1}{2}\right)}\det(\widetilde{\Sigma})^{-1/2}\left\{\tilde{z}^{\mathrm{T}}\widetilde{\Sigma}^{-1}\tilde{z}\right\}^{-\frac{\alpha+k+1}{2}}.$$

Introducing the block decomposition $\widetilde{\Sigma} = \begin{pmatrix} 1 & \Sigma_{0,k} \\ \Sigma_{k,0} & \Sigma_k \end{pmatrix}$, the inverse matrix is

$$\widetilde{\Sigma}^{-1} = \begin{pmatrix} 1 + \Sigma_{0,k}(\Sigma_k - \Sigma_{k,0}\Sigma_{0,k})^{-1}\Sigma_{k,0} & -\Sigma_{0,k}(\Sigma_k - \Sigma_{k,0}\Sigma_{0,k})^{-1} \\ -(\Sigma_k - \Sigma_{k,0}\Sigma_{0,k})^{-1}\Sigma_{k,0} & (\Sigma_k - \Sigma_{k,0}\Sigma_{0,k})^{-1} \end{pmatrix}.$$

By the definition of μ_k and $\widehat{\Sigma}_k$, we have

$$\widetilde{\Sigma}^{-1} = \frac{1}{1+\alpha}\begin{pmatrix} 1 + \alpha + \mu_k^{\mathrm{T}}\widehat{\Sigma}_k^{-1}\mu_k & -\mu_k^{\mathrm{T}}\widehat{\Sigma}_k^{-1} \\ -\widehat{\Sigma}_k^{-1}\mu_k & \widehat{\Sigma}_k^{-1} \end{pmatrix}$$

and

$$\tilde{z}^{\mathrm{T}}\widetilde{\Sigma}^{-1}\tilde{z} = (1,z)^{\mathrm{T}}\widetilde{\Sigma}^{-1}(1,z) = 1 + \frac{(z-\mu_k)^{\mathrm{T}}\widehat{\Sigma}_k^{-1}(z-\mu_k)}{\alpha+1}$$

Finally, we obtain after simplification

$$g(z) = \pi^{-k/2}(\alpha+1)^{-k/2}\frac{\Gamma\left(\frac{k+\alpha+1}{2}\right)}{\Gamma\left(\frac{\alpha+1}{2}\right)}\det(\widehat{\Sigma}_k)^{-1/2}\left\{1 + \frac{(z-\mu_k)^{\mathrm{T}}\widehat{\Sigma}_k^{-1}(z-\mu_k)}{\alpha+1}\right\}^{-\frac{\alpha+k+1}{2}}.$$

We recognize the k-variate Student density with $\alpha+1$ degrees of freedom, location parameter μ_k and scale matrix $\widehat{\Sigma}_k$. □

Proof of Proposition 5. Consider the set

$$A = \{f \in \mathcal{C}_0 :\ f(x_1) \in A_1, \cdots, f(x_k) \in A_k\}.$$

Equations (5) and (14) together with Lemma 2 yield

$$
\begin{aligned}
P_{x_0}(A) &= \int_{\mathcal{C}} 1_{\{f/f(x)\in A\}} f(x)\nu(\mathrm{d}f) = E\left[c_\alpha W(x_0)_+^\alpha 1_{\{W(x_i)_+^\alpha/W(x_0)_+^\alpha\in A_i,\ i=1,\ldots,k\}}\right] \\
&= \widehat{\mathrm{pr}}\left\{W(x_i)_+^\alpha/W(x_0)_+^\alpha \in A_i,\ i=1,\ldots,k\right\} \\
&= \mathrm{pr}\left\{(T_i)_+^\alpha \in A_i,\ i=1,\ldots,k\right\}
\end{aligned}
$$

where $T=(T_1,\ldots,T_k)$ has a multivariate Student distribution with $\alpha+1$ degrees of freedom, location parameter μ_k and dispersion matrix $\widehat{\Sigma}_k$. This proves the result. □

2·4. *Multivariate extreme value distributions*

Proof of Proposition 6. It is easily shown that the logistic model admits the representation

$$Z = \max_{i\geq 1} \zeta_i F_i$$

where the F_i's are independent random vectors with independent $\mathrm{Frechet}(\beta, c_\beta)$-distributed components. To check this, we compute

$$
\begin{aligned}
E\left(\max_{j=1}^{N} \frac{F_j}{z_j}\right) &= \int_0^\infty \mathrm{pr}\left(\max_{j=1}^{N}\frac{F_j}{z_j} > u\right)\mathrm{d}u = \int_0^\infty \left\{1-\prod\nolimits_{j=1}^{N} \mathrm{pr}(F_j < z_j u)\right\}\mathrm{d}u \\
&= \int_0^\infty \left\{1-\prod\nolimits_{j=1}^{N} e^{-(z_j u/c_\beta)^{-\beta}}\right\}\mathrm{d}u = \int_0^\infty \left\{1-e^{-u^{-\beta}\sum_{j=1}^{N}(z_j/c_\beta)^{-\beta}}\right\}\mathrm{d}u \\
&= \left(\sum\nolimits_{j=1}^{N} z_j^{-\beta}\right)^{1/\beta}.
\end{aligned}
$$

For the computation of the last integral, we recognize the expectation of a Fréchet distribution. Next we use the fact that P_{j_0} is the distribution of F/F_{j_0} under the transformed density

$$y_{j_0}\prod_{k=1}^{N}\frac{\beta}{c_\beta}\left(\frac{y_k}{c_\beta}\right)^{-1-\beta} e^{-(y_k/c_\beta)^{-\beta}}.$$

We recognize a product measure where the jth margin, $j\neq j_0$, has a $\mathrm{Frechet}(\beta,c_\beta)$ distribution. The j_0th marginal has density

$$y_{j_0}\frac{\beta}{c_\beta}\left(\frac{y_{j_0}}{c_\beta}\right)^{-1-\beta} e^{-(y_{j_0}/c_\beta)^{-\beta}}$$

and a simple change of variable reveals that this is the density of $c_\beta Z^{-1/\beta}$ with $Z\sim \mathrm{Gamma}(1-1/\beta, 1)$. □

Proof of Proposition 7. Similarly to the logistic model, we have the spectral representation

$$Z = \max_{i\geq 1}\zeta_i W_i$$

where the W_i's are independent random vectors with independent $\mathrm{Weibull}(\theta, c_\theta)$-distributed components with scale parameter $c_\theta = \frac{1}{\Gamma(1+1/\theta)}$. To check this, we compute

$$\begin{aligned}
E\left(\max_{j=1}^{N} \frac{W_j}{z_j}\right) &= \int_0^\infty \mathrm{pr}\left(\max_{j=1}^{N} \frac{W_j}{z_j} > u\right) \mathrm{d}u = \int_0^\infty \left\{1 - \prod_{j=1}^{N} \mathrm{pr}(W_j < z_j u)\right\} \mathrm{d}u \\
&= \int_0^\infty \left[1 - \prod_{j=1}^{N} \left\{1 - e^{-(z_j u / c_\theta)^\theta}\right\}\right] \mathrm{d}u = -\sum_J (-1)^{|J|} \int_0^\infty e^{-u^\theta \sum_{j \in J} (z_j/c_\theta)^\theta} \mathrm{d}u \\
&= -\sum_J (-1)^{|J|} \left\{\sum\nolimits_{j \in J} (z_j/c_\theta)^\theta\right\}^{-1/\theta} \Gamma(1 + 1/\theta) = -\sum_J (-1)^{|J|} \left(\sum\nolimits_{j \in J} z_j^\theta\right)^{-1/\theta}.
\end{aligned}$$

For the computation of the last integral, we recognize the expectation of a Weibull distribution. As for the logistic model, P_{j_0} is the distribution of W/W_{j_0} under the transformed density

$$y_{j_0} \prod_{k=1}^{N} \frac{\theta}{c_\theta} \left(\frac{y_k}{c_\theta}\right)^{\theta-1} e^{-(y_k/c_\theta)^\theta}.$$

We recognize a product measure where the jth margin, $j \neq j_0$, has a $\mathrm{Weibull}(\theta, c_\theta)$ distribution. The j_0th marginal has density

$$y_{j_0} \frac{\theta}{c_\theta} \left(\frac{y_{j_0}}{c_\theta}\right)^{\theta-1} e^{-(y_{j_0}/c_\theta)^\theta}$$

and a simple change of variable reveals that this is the density of $c_\theta Z^{1/\theta}$ with $Z \sim \mathrm{Gamma}(1 + 1/\theta, 1)$. □

Proof of Proposition 8. By definition, P_{j_0} has the form

$$\begin{aligned}
P_{j_0}(A) &= N \sum_{k=1}^{m} \pi_k \int_{S_{N-1}} y_{j_0} 1_{\{y/y_{j_0} \in A\}} \mathrm{diri}(y \mid \alpha_{1k}, \ldots, \alpha_{Nk}) \, \mathrm{d}y \\
&= N \sum_{k=1}^{m} \hat{\pi}_k \frac{\int_{S_{N-1}} y_{j_0} 1_{\{y/y_{j_0} \in A\}} \mathrm{diri}(y \mid \alpha_{1k}, \ldots, \alpha_{Nk}) \, \mathrm{d}y}{\int_{S_{N-1}} y_{j_0} \mathrm{diri}(y \mid \alpha_{1k}, \ldots, \alpha_{Nk}) \, \mathrm{d}y}, \quad A \subset (0, \infty)^N.
\end{aligned}$$

Thus, P_{j_0} is given as the mixture $P_{j_0} = \sum_{k=1}^{m} N \hat{\pi}_k P_{j_0}^{(k)}$, where for each $k = 1, \ldots, m$, the probability measure $P_{j_0}^{(k)}$ is equal to the distribution of the random vector $\tilde{Y}^{(k)}/\tilde{Y}_{j_0}^{(k)}$, and $\tilde{Y}^{(k)}$ has a transformed density proportional to $y_{j_0} \prod_{j=1}^{N} y_j^{\alpha_j - 1}$. We recognize the Dirichlet distribution with parameters $\tilde{\alpha}_{1k}, \ldots, \tilde{\alpha}_{Nk}$ given by

$$\tilde{\alpha}_{j_0 k} = \alpha_{j_0 k} + 1 \quad \text{and} \quad \tilde{\alpha}_{jk} = \alpha_{jk} \quad j \neq j_0.$$

It is well known that Dirichlet distributions can be expressed in terms of Gamma distributions. More precisely, we have the stochastic representation

$$\tilde{Y}^{(k)} = \left(G_1^{(k)} \Big/ \sum_{j=1}^{N} G_j^{(k)}, \ldots, G_N^{(k)} \Big/ \sum_{j=1}^{N} G_j^{(k)}\right),$$

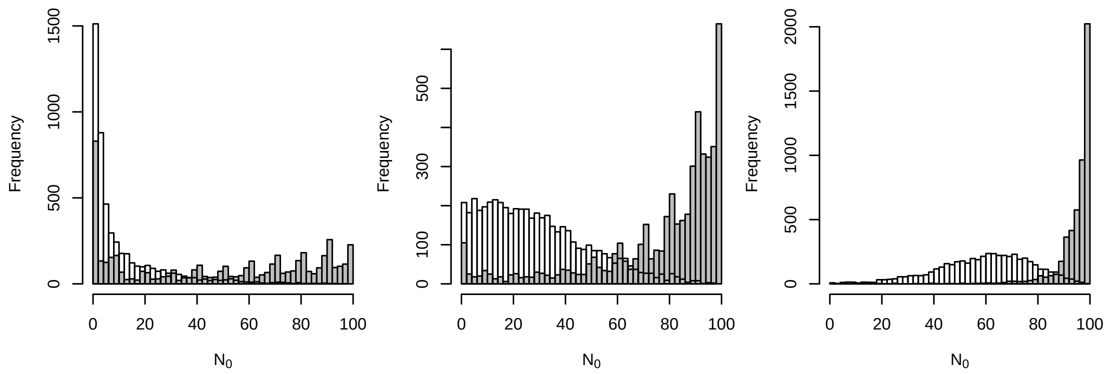

Fig. 1. Histogram of N_0 based on 5000 realizations of a Brown–Resnick process associated to the semi-variogram $\gamma(h) = c\|h\|^\alpha$ with $c = 1$ and $\alpha = 1.5$ (left), $c = 2.5$ and $\alpha = 1$ (middle) and $c = 5$ and $\alpha = 0.5$ simulated via Algorithm 1 with the deterministic design (grey) and the adaptive design (21) (white), respectively.

where $G_j^{(k)}$ are independent $\mathrm{Gamma}(\tilde{\alpha}_{jk}, 1)$ random variables. The result follows since $P_{j_0}^{(k)}$ is the distribution of $\tilde{Y}^{(k)}/\tilde{Y}_{j_0}^{(k)}$. □

3. SIMULATION STUDY

We perform a simulation study to compare the adaptive version of Algorithm 1 introduced in (21) to a version, where the numbering of locations is deterministic. The simulation study is based on 5000 simulations of a Brown–Resnick process associated to a semi-variogram of the type $\gamma(h) = c\|h\|^\alpha$ on the two-dimensional grid $\{0.05, 0.15, \ldots, 0.95\} \times \{0.05, 0.15, \ldots, 0.95\}$. We run Algorithm 1 with the deterministic design (the grid points are ordered by their coordinates in the lexicographical sense) and with the adaptive design (21). The simulation is repeated for different values of the parameter vector (c, α) representing strong dependence ($(c, \alpha) = (1, 1.5)$), moderate dependence ($(c, \alpha) = (2.5, 1)$) and weak dependence ($(c, \alpha) = (5, 0.5)$). The histograms of N_0 are shown in Figure 1. For each of the three parameter vectors, the number N_0 for the adaptive design is stochastically smaller than the corresponding number for the deterministic design.

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