Supplementary Material
Wavelet Spectra for Multivariate Point Processes

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1 Proofs

Where relevant, a and b are such that $(a, b) \in \mathcal{T}_{\alpha,T}$ or $(a, b) \in \mathcal{T}_{\alpha,\kappa,T}$. This allows all integrals over $(0, T)$ to be replaced by integrals over the entire real line. To suppress notation, $\int$ implicitly denotes $\int_{-\infty}^{\infty}$.

We adopt throughout the convention that the Fourier transform $G(f)$ of a function $g(t)$ is defined as $G(f) = \int g(t)\mathrm{e}^{-\mathrm{i}2\pi ft}\mathrm{d}t$, and the inverse Fourier transform defined as $g(t) = \int G(f)\mathrm{e}^{\mathrm{i}2\pi ft}\mathrm{d}f$.

While in the main manuscript it makes sense to present Proposition 1 before Proposition 2, for the purposes of proving these results, it makes sense to prove Proposition 2 first.

Proofs of Propositions 1-3

Proposition 2. *Let $\psi(t)$ satisfy Assumption 1, $h(t)$ satisfy Assumption 2, and for $\kappa > 0$ the corresponding non-negative definite kernel $K(s, t)$ have eigenfunctions $\{\varphi_l(t);\ l = 0, 1, ...\}$ and eigenvalues $\{\eta_l;\ l = 0, 1, ...\}$. It holds that $\sum_l \eta_l |\Phi_l(f)|^2 = |\Psi(f)|^2$ where $\Phi_l(f)$ and $\Psi(f)$ are the Fourier transforms of $\varphi_l(t)$ and $\psi(t)$, respectively.*

Proof. We define

$$\mathcal{K}(f, f') = \int\int K(s, t)\mathrm{e}^{-\mathrm{i}2\pi fs}\mathrm{e}^{-\mathrm{i}2\pi f't}\mathrm{d}s\mathrm{d}t.$$

Using the representation

$$K(s,t) = \int h_\kappa(u)\psi(s-u)\psi^*(t-u)\mathrm{d}u$$

gives $\mathcal{K}(f,-f) = |\Psi(f)|^2$, recalling that $\int_{-\infty}^{\infty} h_\kappa(u)\mathrm{d}u = 1$. Using the representation

$$K(s,t) = \sum_{l=1}^{\infty} \eta_l \varphi_l(s)\varphi_l^*(t)$$

gives $\mathcal{K}(f,-f) = \sum_{l=1}^{\infty} \eta_l |\Phi_l(f)|^2$. The required result follows. □

We can now easily proceed with the proof of Proposition 1.

Proposition 1. *Let $\psi(t)$ satisfy Assumption 1, $h(t)$ satisfy Assumption 2, and for $\kappa > 0$ the corresponding non-negative definite kernel $K(s,t)$ have eigenfunctions $\{\varphi_l(t);\ l = 0, 1, ...\}$. Every eigenfunction $\varphi_l(t)$ with a non-zero eigenvalue is a wavelet that satisfies the conditions of Assumption 1.*

Proof. Each $\varphi_l(t)$ with non-zero eigenvalue will be real (complex) and continuous for real (complex) wavelet $\psi(t)$, by construction. Property (i): from the definition of $K(s,t)$ in (4), it immediately follows that $\int\int K(s,t)\mathrm{d}s\mathrm{d}t = 0$. Furthermore, with $K(s,t) = \sum_{l=0}^{\infty} \eta_l \varphi_l(s)\varphi_l^*(t)$, it follows that

$$\int\int K(s,t)\mathrm{d}s\mathrm{d}t = \sum_{l=0}^{\infty} \eta_l \left| \int \varphi_l(t)\mathrm{d}t \right|^2.$$

Therefore $\left|\int \varphi_l(t)\mathrm{d}t\right| = 0$ for all $\varphi_l(t)$ with positive eigenvalues.

Property (ii) is immediate from the construction of the eigenfunctions.

Property (iii) is immediate from Proposition 2 and the fact that $\psi(t)$ itself obeys the admissibility condition. □

Proposition 3. *Let $N(t)$ be a p-dimensional stationary process with spectral density matrix $S(f)$. Let $\psi(t)$ be a wavelet satisfying Assumption 1 and let $h(t)$ be a smoothing function satisfying Assumption 2. For all $\kappa > 0$ and for all $(\tilde{a}, \tilde{b}) \in \tilde{T}_{\alpha,\kappa}$,*

$$E\{\Omega^{(T)}(\tilde{a},\tilde{b})\} = E\{W^{(T)}(\tilde{a},\tilde{b})\} = \int_{-\infty}^{\infty} \tilde{a}|\Psi^{(T)}(\tilde{a}f)|^2 S(f)\mathrm{d}f$$

and $E\{\Omega^{(T)}(\tilde{a},\tilde{b})\} = S(f_{\tilde{a}}) + O(T^{-2})$ as $T \to \infty$.

Proof. We can write

$$\Omega^{(T)}(\tilde{a},\tilde{b}) = \int_0^T \int_0^T K^{(T)}_{\tilde{a},\tilde{b}}(s,t)\mathrm{d}N(t)\mathrm{d}N^{\mathrm{T}}(s)$$

where

$$K^{(T)}(s,t) = \int h^{(T)}_{\kappa\tilde{a}}(u)\psi^{(T)}(s-u)\psi^{*(T)}(t-u)\mathrm{d}u$$

and $K^{(T)}_{\tilde{a},\tilde{b}}(s,t) = \tilde{a}^{-1}K\{(s-\tilde{b}T)/\tilde{a},(t-\tilde{b}T)/\tilde{a}\}$. If $\{\varphi_l(t); l = 0,1,...\}$ are the eigenfunctions of $K(s,t)$, then $\{\varphi^{(T)}_l(t) = T^{-1/2}\varphi_l(t/T); l = 0,1,...\}$ are the eigenfunctions of $K^{(T)}(s,t)$. It follows that $E\{\Omega^{(T)}(\tilde{a},\tilde{b})\} = \sum_{l=0}^{\infty}\eta_l E\{v^{(T)}_l(\tilde{a},\tilde{b})v^{(T)\mathrm{H}}_l(\tilde{a},\tilde{b})\}$, where $v^{(T)}_l(\tilde{a},\tilde{b}) = \tilde{a}^{-1/2}\int\varphi^{(T)}_l\{(t-\tilde{b}T)/\tilde{a}\}\mathrm{d}N(t)$. Furthermore,

$$\begin{aligned}
E\{v^{(T)}_l(\tilde{a},\tilde{b})v^{(T)\mathrm{H}}_l(\tilde{a},\tilde{b})\} &= \int\int \varphi^{(T)}_l\{(t-\tilde{b}T)/\tilde{a}\}\varphi^{*(T)}_l\{(s-\tilde{b}T)/\tilde{a}\}\Gamma(t,s)\mathrm{d}t\mathrm{d}s \\
&= \int\int \varphi^{(T)}_l\{(t-\tilde{b}T)/\tilde{a}\}\varphi^{*(T)}_l\{(t-\tau-\tilde{b}T)/\tilde{a}\}\Gamma(\tau)\mathrm{d}t\mathrm{d}\tau \\
&= \tilde{a}\int |\Phi^{(T)}_l(\tilde{a}f)|^2 S(f)\mathrm{d}f,
\end{aligned}$$

by using $\varphi^{(T)}(t) = \int\Phi^{(T)}(f)\mathrm{e}^{\mathrm{i}2\pi ft}\mathrm{d}f$. Therefore, from Proposition 2

$$E\{\Omega^{(T)}(\tilde{a},\tilde{b})\} = E\{W^{(T)}(\tilde{a},\tilde{b})\} = \tilde{a}\int|\Psi^{(T)}(\tilde{a}f)|^2 S(f)\mathrm{d}f.$$

Taking the Taylor expansion of $S(f)$ around $f_{\tilde{a}}$ gives

$$\begin{aligned}
\tilde{a}\int|\Psi^{(T)}(\tilde{a}f)|^2S(f)\mathrm{d}f &= \tilde{a}\int|\Psi^{(T)}(\tilde{a}f)|^2\{S(f_{\tilde{a}}) + (f-f_{\tilde{a}})S'(f_{\tilde{a}}) + \frac{(f-f_{\tilde{a}})^2}{2}S''(f_{\tilde{a}}) + ...\}\mathrm{d}f \\
&= S(f_{\tilde{a}})\tilde{a}\int|\Psi^{(T)}(\tilde{a}f)|^2\mathrm{d}f + \\
&\qquad S'(f_{\tilde{a}})\left\{\tilde{a}\int f|\Psi^{(T)}(\tilde{a}f)|^2\mathrm{d}f - f_{\tilde{a}}\tilde{a}\int|\Psi^{(T)}(\tilde{a}f)|^2\mathrm{d}f\right\} + \\
&\qquad \frac{S''(f_{\tilde{a}})}{2}\tilde{a}\int(f-f_{\tilde{a}})^2|\Psi^{(T)}(\tilde{a}f)|^2\mathrm{d}f + ... \\
&= S(f_{\tilde{a}}) + \frac{S''(f_{\tilde{a}})}{2}\tilde{a}\int|\Psi^{(T)}(\tilde{a}f)|^2(f-f_{\tilde{a}})^2\mathrm{d}f + ... \\
&= S(f_{\tilde{a}}) + \frac{S''(f_{\tilde{a}})\sigma^2_{\psi^{(T)}}}{2\ \tilde{a}^2} + ...
\end{aligned}$$

The result follows because $\sigma^2_{\psi^{(T)}}$ is $O(T^{-2})$. □

Lemmas for Theorems 1-2

The following Lemma is presented as Corollary 3.1 in Brillinger (1972).

Lemma 1. *Let $N(t)$ be a p-dimensional point process satisfying Assumption 3, and let $\xi_1(t), ..., \xi_k(t)$ be continuous functions with finite support, then*

$$\text{cum}\left\{\int \xi_1(t_1)\text{d}N_{i_1}(t_1), ..., \int \xi_k(t_k)\text{d}N_{i_k}(t_k)\right\} = \sum_{l=1}^{k} \sum_{\alpha_1,...,\alpha_l=1}^{p} \left(\prod_{j\in v_1} \delta_{\alpha_1 i_j}\right) \cdots \left(\prod_{j\in v_l} \delta_{\alpha_l i_j}\right)$$
$$\times \int \cdots \int \left\{\prod_{j\in v_1} \xi_j(\tau_1)\right\} \cdots \left\{\prod_{j\in v_l} \xi_j(\tau_l)\right\} q_{\alpha_1,...,\alpha_l}(\tau_1, ..., \tau_l)\text{d}\tau_1, .., \text{d}\tau_l, \quad \text{(S1)}$$

where $\delta_{ij} = 1$ if $i = j$ and is zero otherwise.

The first summation in (S1) does not have just k terms, but instead extends over all partitions of $\{1, ..., k\}$ of the form $(v_1, ..., v_l)$. For example, for $k = 3$, the $l = 1$ partition is $(\{1, 2, 3\})$, the $l = 2$ partitions are $(\{1\}, \{2, 3\})$, $(\{2\}, \{1, 3\})$ and $(\{3\}, \{1, 2\})$, and the $l = 3$ partition is $(\{1\}, \{2\}, \{3\})$, resulting in 5 terms. To proceed, we will also require the following lemma.

Lemma 2. *Let $N(t)$ be a p-dimensional point process satisfying Assumption 3, and let $\psi_1(t)$ and $\psi_2(t)$ be a pair of orthogonal wavelets, each satisfying Assumption 4. The cumulant*

$$\text{cum}\left\{\int \psi_1^{(T)}(t_1)\text{d}N_{i_1}(t_1), \int \psi_2^{*(T)}(t_2)\text{d}N_{i_2}(t_2)\right\}$$

is $O(T^{-1})$.

Proof. From Lemma 1, the cumulant equals

$$\int\int \psi_1^{(T)}(t+u)\psi_2^{*(T)}(t)r_{i_1,i_2}(u)\text{d}u\text{d}t.$$

The stated orthogonality of $\psi_1(t)$ and $\psi_2(t)$ implies $\int \psi_1^{(T)}(t)\psi_2^{*(T)}(t)\text{d}t = 0$ and Assumption

4 gives $\int \left|\psi_1^{(T)}(t+u) - \psi_1^{(T)}(t)\right| \mathrm{d}t \leq T^{-1/2}C|u|$. Therefore,

$$\begin{aligned}
&\left|\int\int \psi_1^{(T)}(t+u)\psi_2^{*(T)}(t) r_{i_1,i_2}(u)\mathrm{d}u\mathrm{d}t\right| \\
&\quad = \left|\int\int \psi_1^{(T)}(t+u)\psi_2^{*(T)}(t) r_{i_1,i_2}(u)\mathrm{d}u\mathrm{d}t - \int\int \psi_1^{(T)}(t)\psi_2^{*(T)}(t) r_{i_1,i_2}(u)\mathrm{d}u\mathrm{d}t.\right| \\
&\quad \leq \int\int \left|\psi_1^{(T)}(t+u)\psi_2^{*(T)}(t) - \psi_1^{(T)}(t)\psi_2^{*(T)}(t)\right| |r_{i_1,i_2}(u)|\mathrm{d}u\mathrm{d}t \\
&\quad \leq \int\int \left|\psi_2^{(T)}(t)\right| \left|\psi_1^{(T)}(t+u) - \psi_1^{(T)}(t)\right| |r_{i_1,i_2}(u)|\mathrm{d}u\mathrm{d}t \\
&\quad \leq T^{-1/2}A\int\int \left|\psi_1^{(T)}(t+u) - \psi_1^{(T)}(t)\right| |r_{i_1,i_2}(u)|\mathrm{d}u\mathrm{d}t \leq T^{-1}AC\int |u||r_{i_1,i_2}(u)|\mathrm{d}u,
\end{aligned}$$

where $A \equiv \max_t |\psi_2(t)|$. Hence, by Assumption 3, the given cumulant is $O(T^{-1})$. □

Lemma 3. *Let $\psi(t)$ be a complex valued wavelet satisfying Assumption 4, and $h(t)$ a smoothing function satisfying Assumption 5. For $\kappa > 0$, the eigen-wavelets of corresponding kernel $K(s,t)$ also satisfy Assumption 4.*

Proof. The variability condition follows trivially from the variability condition on $\psi(t)$. The orthogonality condition also follows from the orthogonality condition on $\psi(t)$. Specifically, it is true that $\int\int K(s,t)K(s,t)\mathrm{d}s\mathrm{d}t = 0$. Furthermore,

$$\int\int K(s,t)K(s,t)\mathrm{d}s\mathrm{d}t = \sum_{l=0}^{\infty} \eta_l^2 \left\{\int \varphi_l(t)\varphi_l(t)\mathrm{d}t\right\}^2.$$

Therefore, $\int \varphi_l(t)\varphi_l(t)\mathrm{d}t = 0$ for $l = 0, 1, ...,$ and $\varphi_l(t)$ is orthogonal to its complex conjugate. □

Proof of Theorem 1 and 2, and Proposition 4

Theorem 1. *Let $N(t)$ be a p-dimensional stationary process satisfying Assumption 3 with spectral density matrix $S(f)$, and let $\psi(t)$ be a wavelet with central frequency f_0 satisfying Assumption 4. The continuous wavelet transform $w^{(T)}(\tilde{a},\tilde{b})$ is asymptotically $\mathcal{N}_p^{\mathcal{C}}\{0, S(f_{\tilde{a}})\}$ as $T \to \infty$, for all $(\tilde{a},\tilde{b}) \in \tilde{\mathcal{T}}_{\alpha,\kappa}$.*

Proof. We first verify the mean and covariance of $w^{(T)}(\tilde{a},\tilde{b})$ are as given. The mean is

$$\begin{aligned}E\{w^{(T)}(\tilde{a},\tilde{b})\} &= E\left[\int_0^T \psi^{*(T)}\{(t-\tilde{b}T)/\tilde{a}\}\mathrm{d}N(t)\right]\\ &= \int_0^T \psi^{*(T)}\{(t-\tilde{b}T)/\tilde{a}\}E\{\mathrm{d}N(t)\}\\ &= \int_0^T \psi^{*(T)}\{(t-\tilde{b}T)/\tilde{a}\}\lambda(t)\mathrm{d}t.\end{aligned}$$

Under the assumptions of the theorem, $N(t)$ is stationary, hence $\lambda(t)$ is constant for all t and $E\{w(\tilde{a},\tilde{b})\} = 0$ as the wavelet integrates to zero. The asymptotic result for $\mathrm{cov}\{w^{(T)}(\tilde{a},\tilde{b})\} = E\{W^{(T)}(\tilde{a},\tilde{b})\}$ is given in Proposition 3.

Additionally, we are required to show $\mathrm{cov}\{w^{(T)}(\tilde{a},\tilde{b}), w^{*(T)}(\tilde{a},\tilde{b})\}$ is asymptotically zero for a circular complex normal distribution. We note for a pair of complex random variables W and Z, $\mathrm{cov}(W, Z^*) = \mathrm{cum}(W, Z)$. Therefore,

$$\begin{aligned}&\mathrm{cov}\{w_{i_1}^{(T)}(\tilde{a},\tilde{b}), w_{i_2}^{*(T)}(\tilde{a},\tilde{b})\}\\ &\quad= \mathrm{cum}\left\{\tilde{a}^{-1/2}\int \psi^{*(T)}\left(\frac{t_1-\tilde{b}T}{\tilde{a}}\right)\mathrm{d}N_{i_1}(t_1), \tilde{a}^{-1/2}\int \psi^{*(T)}\left(\frac{t_2-\tilde{b}T}{\tilde{a}}\right)\mathrm{d}N_{i_2}(t_2)\right\}.\end{aligned}$$

Assumption 4 implies $\psi^{(T)}(t)$ is orthogonal to $\psi^{*(T)}(t)$. Setting $\psi_1(t) = \psi^{*(T)}(t)$ and $\psi_2(t) = \psi^{(T)}(t)$ in Lemma 2 gives all the terms in $\mathrm{cov}\{w^{(T)}(\tilde{a},\tilde{b}), w^{*(T)}(\tilde{a},\tilde{b})\}$ as $O(T^{-1})$. All first and second order cumulants are therefore asymptotically equal to those stated in the theorem.

To conclude, we are required to show that all cumulants of order greater than two asymptotically go to zero. These cumulants can be written in the form

$$\mathrm{cum}\{w_{i_1}^{(T)}(\tilde{a},\tilde{b}), ..., w_{i_{k'}}^{(T)}(\tilde{a},\tilde{b}), w_{i_{k'+1}}^{*(T)}(\tilde{a},\tilde{b}), ..., w_{i_k}^{*(T)}(\tilde{a},\tilde{b})\},$$

where $0 \le k' \le k$, $k > 2$. In the quest to simplify notation, we present the $k' = 0$ case, i.e. cumulants of the form $\mathrm{cum}\{w_{i_1}^{*(T)}(\tilde{a},\tilde{b}), ..., w_{i_k}^{*(T)}(\tilde{a},\tilde{b})\}$. The extension to cumulants that include both forms of conjugation is trivial.

From Lemma 1,

$$\begin{aligned}&\mathrm{cum}\{w_{i_1}^{*(T)}(\tilde{a},\tilde{b}), ..., w_{i_k}^{*(T)}(\tilde{a},\tilde{b})\} = \sum_{l=1}^{k}\sum_{\alpha_1,\ldots,\alpha_l=1}^{p}\left(\prod_{j\in v_1}\delta_{\alpha_1 i_j}\right)\cdots\left(\prod_{j\in v_l}\delta_{\alpha_l i_j}\right)\\ &\times\int\cdots\int\left[\tilde{a}^{-1/2}\psi^{(T)}\{(\tau_1-\tilde{b}T)/\tilde{a}\}\right]^{|v_1|}\cdots[\tilde{a}^{-1/2}\psi^{(T)}\{(\tau_l-\tilde{b}T)/\tilde{a}\}]^{|v_l|}q_{\alpha_1,\ldots,\alpha_l}(\tau_1,...,\tau_l)\mathrm{d}\tau_1,..,\mathrm{d}\tau_l.\end{aligned}$$

Through a change of variables and under Assumption 3 it follows that

$$\text{cum}\{w_{i_1}^{*(T)}(\tilde{a},\tilde{b}),...,w_{i_k}^{*(T)}(\tilde{a},\tilde{b})\} = \sum_{l=1}^{k}\sum_{\alpha_1,...,\alpha_l=1}^{p}\left(\prod_{j\in v_1}\delta_{\alpha_1 i_j}\right)\cdots\left(\prod_{j\in v_l}\delta_{\alpha_l i_j}\right)$$
$$\int\cdots\int\left[\tilde{a}^{-1/2}\psi^{(T)}\{(t+u_1)/\tilde{a}\}\right]^{|v_1|}\cdots\left[\tilde{a}^{-1/2}\psi^{(T)}\{(t+u_{l-1})/\tilde{a}\}\right]^{|v_{l-1}|}\left[\tilde{a}^{-1/2}\psi^{(T)}(t/\tilde{a})\right]^{|v_l|}$$
$$\cdot\, r_{\alpha_1,...,\alpha_l}(u_1,...,u_{l-1})\mathrm{d}u_1,..,\mathrm{d}u_{l-1}\mathrm{d}t.$$

Recognising that the product of Kronecker deltas is either zero or one gives

$$\text{cum}\{w_{i_1}^{*(T)}(\tilde{a},\tilde{b}),...,w_{i_k}^{*(T)}(\tilde{a},\tilde{b})\} \leq \sum_{l=1}^{k}\sum_{\alpha_1,...,\alpha_l=1}^{p}$$
$$\int\cdots\int\left[\tilde{a}^{-1/2}\psi^{(T)}\{(t+u_1)/\tilde{a}\}\right]^{|v_1|}\cdots\left[\tilde{a}^{-1/2}\psi^{(T)}\{(t+u_{l-1})/\tilde{a}\}\right]^{|v_{l-1}|}\left\{\tilde{a}^{-1/2}\psi^{(T)}(t/\tilde{a})\right\}^{|v_l|}$$
$$\cdot\, r_{\alpha_1,...,\alpha_l}(u_1,...,u_{l-1})\mathrm{d}u_1,..,\mathrm{d}u_{l-1}\mathrm{d}t. \quad \text{(S2)}$$

Using the fact that $\sum_{j=1}^{l}|v_j| = k$, Hölder's inequality gives

$$\int\left|\left[\psi^{(T)}(t+u_1)\right]^{|v_1|}\cdots\left[\psi^{(T)}(t+u_{l-1})\right]^{|v_{l-1}|}\left[\psi^{(T)}(t)\right]^{|v_l|}\right|\mathrm{d}t$$
$$\leq\left(\int\left|\psi^{(T)}(t)\right|^k\mathrm{d}t\right)^{|v_l|/k}\prod_{\beta=1}^{l-1}\left(\int\left|\psi^{(T)}(t+u_l)\right|^k\mathrm{d}t\right)^{|v_\beta|/k} = \int\left|\psi^{(T)}(t)\right|^k\mathrm{d}t = A_kT^{1-k/2} \quad \text{(S3)}$$

where $A_k = \int|\psi(t)|^k\mathrm{d}t$. Putting (S2) and (S3) together, if follows that

$$|\text{cum}\{w_{i_1}^{*(T)}(\tilde{a},\tilde{b}),...,w_{i_k}^{*(T)}(\tilde{a},\tilde{b})\}|$$
$$\leq(\tilde{a}T)^{1-k/2}A_k\sum_{l=1}^{k}\sum_{\alpha_1,...,\alpha_l=1}^{p}\int\cdots\int\left|r_{\alpha_1,...,\alpha_l}(u_1,...,u_{l-1})\right|\mathrm{d}u_1,..,\mathrm{d}u_{l-1}. \quad \text{(S4)}$$

Therefore, from Assumption 3, $\text{cum}\{w_{i_1}^{*(T)}(\tilde{a},\tilde{b}),...,w_{i_k}^{*(T)}(\tilde{a},\tilde{b})\}$ is $O(T^{1-k/2})$ and tends to zero as $T\to\infty$ for all $k>2$. Therefore all cumulants of order greater than two asymptotically go to zero, giving the required asymptotic normality. □

Theorem 2. *Let $N(t)$ be a p-dimensional stationary process satisfying Assumption 3 with spectral density matrix $S(f)$. Let $\psi(t)$ be a wavelet with central frequency f_0 satisfying Assumption 4, let $h(t)$ be a smoothing function satisfying Assumption5, and for $\kappa>0$ let*

$\{\eta_l;\ l=0,1,...\}$ *be the eigenvalues of the kernel* $K(s,t)$ *defined in (4). The temporally smoothed wavelet periodogram* $\Omega^{(T)}(\tilde{a},\tilde{b})$ *is asymptotically* $(1/n)\mathcal{W}_p^{\mathcal{C}}\{n,S(f_{\tilde{a}})\}$ *as* $T\to\infty$ *for all* $(\tilde{a},\tilde{b})\in\tilde{\mathcal{T}}_{\alpha,\kappa}$*, where* $n=1/\left(\sum_{l=1}^{\infty}\eta_l^2\right)$.

Proof. Consider the eigenwavelet representation of the temporally smoothed wavelet periodogram in (5). By Lemma 3 and Theorem 1, $v_l^{(T)}(a,b)$ is asymptotically $\mathcal{N}\{0,S(f_0/\tilde{a})\}$, $l=0,1,....$ Therefore, $v_l^{(T)}(\tilde{a},\tilde{b})v_l^{(T)}(\tilde{a},\tilde{b})^{\mathrm{H}}$ is asymptotically $\mathcal{W}_p^{\mathcal{C}}\{1,S(f_0/\tilde{a})\}$. As the eigenwavelet system is orthonormal, Lemma 2 states that $v_0^{(T)}(\tilde{a},\tilde{b}),v_1^{(T)}(\tilde{a},\tilde{b}),...$ are asymptotically independent random vectors and therefore it follows in an analogous manner to (Walden, 2000, p. 776) that $\Omega^{(T)}(\tilde{a},\tilde{b})$ is asymptotically $(1/n)\mathcal{W}_p^{\mathcal{C}}\{n,S(f_0/\tilde{a})\}$ where $n=1/\sum_{l=0}^{\infty}\eta_l^2$. □

Proposition 4. *Let* $\psi(t)$ *satisfying Assumption 4, let* $h(t)$ *be the rectangular smoothing window given in (6), and for* $\kappa>0$ *let corresponding kernel* $K(s,t)$ *have ordered eigenvalues* $\{\eta_l;\ l=0,1,...\}$*. Provided* $\kappa>\alpha$*, then* $n=(\sum_{l=0}^{\infty}\eta_l^2)^{-1}=\kappa\left\{\int_{-\infty}^{\infty}|\mathcal{P}(x)|^2\mathrm{d}x\right\}^{-1}$*, where* $\mathcal{P}(x)\equiv\int_{-\infty}^{\infty}\psi(t)\psi^*(t-x)\mathrm{d}t$.

Proof. It holds that

$$\int\int K(s,t)K^*(s,t)\mathrm{d}s\mathrm{d}t=\int\int\left\{\sum_{l=0}^{\infty}\eta_l\varphi_l(s)\varphi_l^*(t)\right\}\left\{\sum_{l'=0}^{\infty}\eta_{l'}\varphi_{l'}^*(s)\varphi_{l'}(t)\right\}\mathrm{d}s\mathrm{d}t=\sum_{l=0}^{\infty}\eta_l^2 \tag{S5}$$

by the orthogonality of the eigenfunctions $\{\varphi_l(t);\ l=0,1,...\}$. Now,

$$\begin{aligned}\int\int K(s,t)K^*(s,t)\mathrm{d}s\mathrm{d}t&=\int\int\int\int h_\kappa(u)h_\kappa(v)\psi(s-u)\psi^*(t-u)\psi^*(s-v)\psi(t-v)\mathrm{d}s\mathrm{d}t\mathrm{d}u\mathrm{d}v\\&=\frac{1}{\kappa^2}\int_{-\kappa/2}^{\kappa/2}\int_{-\kappa/2}^{\kappa/2}\int\int\psi(s-u)\psi^*(s-v)\psi^*(t-u)\psi(t-v)\mathrm{d}s\mathrm{d}t\mathrm{d}u\mathrm{d}v.\end{aligned}$$

Considering individual integrals, we have

$$\int\psi(s-u)\psi^*(s-v)\mathrm{d}s=\int\psi(s)\psi^*(s-(v-u))\mathrm{d}s=\mathcal{P}(v-u)$$

where $\mathcal{P}(x) \equiv \int \psi(t)\psi^*(t-x)\mathrm{d}t$. Therefore

$$\int\int K(s,t)K^*(s,t)\mathrm{d}s\mathrm{d}t = \frac{1}{\kappa^2}\int_{-\kappa/2}^{\kappa/2}\int_{-\kappa/2}^{\kappa/2}|\mathcal{P}(v-u)|^2\mathrm{d}u\mathrm{d}v$$
$$= \frac{1}{2\kappa^2}\int_{-\kappa}^{\kappa}\int_{-\kappa}^{\kappa}|\mathcal{P}(x)|^2\mathrm{d}x\mathrm{d}y$$
$$= \frac{1}{\kappa}\int_{-\kappa}^{\kappa}|\mathcal{P}(x)|^2\mathrm{d}x.$$

The (approximating) support of $\psi(t)$ is $(-\alpha/2, \alpha/2)$, therefore the (approximating) support of $|\mathcal{P}(x)|^2$ is $(-\alpha, \alpha)$. Hence, provided $\alpha < \kappa$, it follows from (S5) that

$$n = \left(\sum_{l=0}^{\infty}\eta_l^2\right)^{-1} = \kappa\left\{\int|\mathcal{P}(x)|^2\mathrm{d}x\right\}^{-1}.$$

□

Test for non-stationarity:

Proofs of Propositions 5-7, and Theorem 3.

Proposition 5. *Let $B_1, ..., B_K$ be independent samples where $B_i \sim (1/n)\mathcal{W}_p^{\mathcal{C}}(n, \Sigma_i)$ ($i = 1, ..., K$). The likelihood ratio test statistic for the null hypothesis $H : \Sigma_1 = ... = \Sigma_K = \Sigma$, with unspecified Σ, is*

$$U(K) = K^{pKn}\frac{\prod_{i=1}^{K}\det(B_i)^n}{\det\left(\sum_{i=1}^{K}B_i\right)^{Kn}}.$$

Furthermore, when H is true, $-2\log(U(K))$ is asymptotically χ_f^2 as $n \to \infty$, where $f = (K-1)p^2$.

Proof. The probability density function for $B_i \sim \mathcal{W}_p^{\mathcal{C}}(n, \Sigma_i)$ is given as

$$f(B_i) = \frac{\det(B_i)^{n-p}\mathrm{e}^{-\mathrm{tr}(\Sigma_i^{-1}B_i)}}{\det(\Sigma_i)^n\Gamma_p^{\mathcal{C}}(n)},$$

where $\Gamma_p^{\mathcal{C}}(\cdot)$ is the complex multivariate Gamma function. Apart from the multiplicative constant, the likelihood function based on the independent samples $B_1, ..., B_K$ is

$$L(\Sigma_1, ..., \Sigma_K) = \prod_{i=1}^{K}\frac{\det(B_i)^{n-p}\mathrm{e}^{-\mathrm{tr}(\Sigma_i^{-1}B_i)}}{\det(\Sigma_i)^n},$$

and the likelihood ratio test statistic is defined as

$$U(K) \equiv \frac{\sup_{\Sigma>0} L(\Sigma, ..., \Sigma)}{\sup_{\Sigma_1,...,\Sigma_K>0} L(\Sigma_1, ..., \Sigma_K)} = \frac{L(n^{-1}\bar{B}, ..., n^{-1}\bar{B})}{L(n^{-1}B_1, ..., n^{-1}B_K)}, \tag{S6}$$

where $\bar{B} = (1/K)\sum_{i=1}^{K} B_i$. Hence the numerator in (S6) is

$$\begin{aligned} L(n^{-1}\bar{B}, ..., n^{-1}\bar{B}) &= \prod_{i=1}^{K} \frac{\det(n^{-1}B_i)^{n-p} \mathrm{e}^{-\mathrm{tr}(n\bar{B}^{-1}B_i)}}{\det(n^{-1}\bar{B})^n} \\ &= \frac{(nK)^{pnK}\mathrm{e}^{-pnK}}{\det\left(\sum_{i=1}^{K} B_i\right)^{nK}} \prod_{i=1}^{K} \det(n^{-1}B_i)^{n-p}, \end{aligned}$$

and the denominator is

$$\begin{aligned} L(n^{-1}B_1, ..., n^{-1}B_K) &= \prod_{i=1}^{K} \frac{\det(n^{-1}B_i)^{n-p} \mathrm{e}^{-\mathrm{tr}(nB_i^{-1}B_i)}}{\det(n^{-1}B_i)^n} \\ &= n^{pnK}\mathrm{e}^{-pnK} \prod_{i=1}^{K} \frac{\det(n^{-1}B_i)^{n-p}}{\det\left(\sum_{i=1}^{K} B_i\right)^{n}}. \end{aligned}$$

Therefore

$$U(K) = K^{pKn} \frac{\prod_{i=1}^{K} \det(B_i)^n}{\det\left(\sum_{i=1}^{K} B_i\right)^{Kn}}.$$

The asymptotic distribution follows from the general theory of likelihood ratio tests. □

In the following proposition, we let $\tilde{U}_j$ be a random variable that is equal in distribution to $U(2^j)$ under the null hypothesis.

Proposition 6. *Let Assumptions 3, 4, 5 and 6 be true, and for a fixed $j \in \{1, ..., J\}$ define the test statistic for H_j as*

$$V_j = K^{pKn} \frac{\prod_{i=1}^{K} \det\{\Omega^{(T)}(\tilde{a}_j, \tilde{b}_i)\}^n}{\det\left\{\sum_{i=1}^{K} \Omega^{(T)}(\tilde{a}_j, \tilde{b}_i)\right\}^{Kn}},$$

where $K = 2^j$ and n is as given in Theorem 2. Under H_j, $V_j \overset{\mathrm{d}}{=} \tilde{U}_j + o(1)$ as $T \to \infty$.

Proof. We consider the eigen-wavelet decomposition

$$\Omega^{(T)}(\tilde{a}, \tilde{b}) = \sum_{k=0}^{\infty} \eta_k v_k^{(T)}(\tilde{a}, \tilde{b}) v_k^{(T)}(\tilde{a}, \tilde{b})^{\mathrm{H}},$$

with $\sum_{k=0}^{\infty} \eta_k = 1$. Letting $\Sigma = E\{\Omega(\tilde{a}, \tilde{b})\}$, it follows from Theorem 1 and the Cornish-Fisher inversion (Barndorff-Nielsen & Cox, 1989, p. 117) that $\nu_k^{(T)}(\tilde{a}, \tilde{b}) \stackrel{\mathrm{d}}{=} Z_k + R_k^{(T)}$, where Z_k are iid $N_p^{\mathcal{C}}(0, \Sigma)$ and $R_k^{(T)} = O(T^{-1/2})$. Therefore,

$$\begin{aligned}
\Omega^{(T)}(\tilde{a}, \tilde{b}) &\stackrel{\mathrm{d}}{=} \sum_{k=0}^{\infty} \eta_k Z_k Z_k^{\mathrm{H}} + \sum_{k=0}^{\infty} \eta_k Z_k R_k^{(T)\mathrm{H}} + \sum_{k=0}^{\infty} \eta_k R_k^{(T)} Z_k^{\mathrm{H}} + \sum_{k=0}^{\infty} \eta_k R_k^{(T)} R_k^{(T)\mathrm{H}} \\
&\stackrel{\mathrm{d}}{=} B + O(T^{-1/2}) + O(T^{-1/2}) + O(T^{-1}). \\
&\stackrel{\mathrm{d}}{=} B + O(T^{-1/2}).
\end{aligned}$$

where $B \sim (1/n)\mathcal{W}_p^{\mathcal{C}}(n, \Sigma)$. Furthermore, it is true that $\det\{\Omega^{(T)}(\tilde{a}, \tilde{b})\} \stackrel{\mathrm{d}}{=} \det(B) + O(T^{-1/2})$. Let $\Omega_{j,i}^{(T)}$ be shorthand for $\Omega^{(T)}(\tilde{a}_j, \tilde{b}_i)$. The kernels centred at $(\tilde{a}_j, \tilde{b}_{j,l})$ and $(\tilde{a}_j, \tilde{b}_{j,m})$ $(l \neq m)$ are non-overlapping by construction of the dyadic partition. Hence, the eigen-wavelets at $(\tilde{a}_j, \tilde{b}_{j,l})$ are orthogonal to the eigenwavelets at $(\tilde{a}_j, \tilde{b}_{j,m})$. Lemma 2 combined with a similar argument to Theorem 1 leads to $\Omega^{(T)}(\tilde{a}_j, \tilde{b}_{j,1}), ..., \Omega^{(T)}(\tilde{a}_j, \tilde{b}_{j,2^j})$ being asymptotically independent as $T \to \infty$ for fixed $j \in \{1, ..., J\}$. Let $B_1, ..., B_K$ be independent samples where $B_i \sim (1/n)\mathcal{W}_p^{\mathcal{C}}(n, \Sigma_i)$ $(i = 1, ..., K)$. It follows that under H_j,

$$\begin{aligned}
\log(V_j) &= pKn\log(K) + n\sum_{i=1}^{K} \log\left\{\det\left(\Omega_{j,i}^{(T)}\right)\right\} - Kn\log\left\{\det\left(\sum_{i=1}^{K} \Omega_{j,i}^{(T)}\right)\right\} \\
&\stackrel{\mathrm{d}}{=} pKn\log(K) + n\sum_{i=1}^{K} \log\left\{\det(B_i) + O(T^{-1/2})\right\} + Kn\log\left\{\det\left(\sum_{i=1}^{K} B_i\right) + O(T^{-1/2})\right\} \\
&\stackrel{\mathrm{d}}{=} pKn\log(K) + n\sum_{i=1}^{K} \left[\log\{\det(B)\} + O(T^{-1/2})\right] + Kn\left[\log\left\{\det\left(\sum_{i=1}^{K} B_i\right)\right\} + O(T^{-1/2})\right].
\end{aligned}$$

From Proposition 4, if $\kappa = O(T^c)$, then so too is $n = O(T^c)$. Therefore

$$\begin{aligned}
\log(V_j) &\stackrel{\mathrm{d}}{=} pKn\log(K) + n\sum_{i=1}^{K} \log\{\det(B_i)\} + Kn\log\left\{\det\left(\sum_{i=1}^{K} B_i\right)\right\} + O(KT^{c-1/2}) \\
&\stackrel{\mathrm{d}}{=} \log(\tilde{U}_j) + O(2^j T^{c-1/2}).
\end{aligned}$$

Therefore

$$V_j \stackrel{\mathrm{d}}{=} \tilde{U}_j\{1 + O(2^j T^{c-1/2})\}, \tag{S7}$$

and hence $V_j \stackrel{\mathrm{d}}{=} \tilde{U}_j + o(1)$ if $0 < c < 1/2$. □

Theorem 3. *Let Assumptions 3, 4, 5 and 6 be true. Under H_j $(j = 1, ..., J)$,* $\mathrm{pr}\{-2\log(V_j) \leq x\} = \mathrm{pr}(\chi^2_{\nu_j} \leq x) + o(1)$ *as $T \to \infty$, where $\nu_j = (2^j - 1)p^2$.*

Proof. It is true that $E(V_j^k) = E[\tilde{U}_j^k\{1 + O(2^j T^{c-1/2})\}^k] = E[\tilde{U}_j^k\{1 + O(2^j T^{c-1/2})\}]$. Let $Y_j = -2\log(V_j)$, then the characteristic function of Y_j is given as

$$\phi_{Y_j}(t) = E(\mathrm{e}^{\mathrm{i}tY_j}) = E\{\mathrm{e}^{-2\mathrm{i}t\log(V_j)}\}. \tag{S8}$$

From Equation (S7), we have

$$\phi_{Y_j}(t) = E(\tilde{U}_j^{-2\mathrm{i}t})\{1 + O(2^j T^{c-1/2})\} = \phi_{\tilde{Y}_j}(t)\{1 + O(2^j T^{c-1/2})\},$$

where $\tilde{Y}_j = -2\log(\tilde{U}_j)$. From (Muirhead, 1985, p. 306), $\phi_{\tilde{Y}_j}(t) = (1-2\mathrm{i}t)^{-\nu_j/2}\{1+O(T^{-c})\}$, and hence

$$\begin{aligned}\phi_{Y_j}(t) &= (1 - 2\mathrm{i}t)^{-\nu_j/2}\{1 + O(T^{-c})\}\{1 + O(2^j T^{c-1/2})\}\\ &= (1 - 2\mathrm{i}t)^{-\nu_j/2}\{1 + O(T^{-c}) + O(2^j T^{c-1/2}) + O(2^j T^{-1/2})\}.\end{aligned}$$

Therefore, for $0 < c < 1/2$, it follows that $\phi_{Y_j}(t) = (1-2\mathrm{i}t)^{-\nu_j/2}\{1+O(T^{-c})+O(2^j T^{c-1/2})\}$. Therefore, for fixed j, $\mathrm{pr}\{-2\log(V_j) \le x\} = \mathrm{pr}(\chi^2_{\nu_j} \le x) + o(1)$ as $T \to \infty$.

□

Corollary 1. *Let Assumptions 3, 4, 5, 6 and 7 be true. Under* H_0, $\mathrm{pr}\{-2\log(V) \le x\} = \mathrm{pr}(\chi^2_\nu \le x) + o(1)$ *as* $T \to \infty$, *where* $\nu = \sum_{j=1}^J \nu_j = p^2(2^{J+1} - 2 - J)$.

Proof. Assumption 7 ensures asymptotic independence of $V_1, ..., V_J$. It follows that asymptotically $\phi_V(t) = \prod_{j=1}^J \phi_{V_j}(t) = (1 - 2it)^{-\nu/2}\{1 + O(T^{-c}) + O(2^J T^{c-1/2})\}$. Assumption 6 gives $\mathrm{pr}\{-2\log(V) \le x\} = \mathrm{pr}(\chi^2_\nu \le x) + o(1)$ as $T \to \infty$. □

2 Real valued wavelets

The results for real valued wavelets are extremely similar to the complex valued wavelet setting. Specifically, a similar line of argument results in $w^{(T)}(\tilde{a}, \tilde{b})$ being asymptotically $\mathcal{N}_p\{0, S(f_{\tilde{a}})\}$ as $T \to \infty$, for all $(\tilde{a}, \tilde{b}) \in \tilde{\mathcal{T}}_{\alpha,\kappa}$. The condition placed on complex-valued wavelets to be orthogonal to their complex conjugate clearly does not hold for real valued wavelets, however it is only needed in the complex setting to obtain a circular complex-Gaussian distribution, which is not needed for the real valued wavelet.

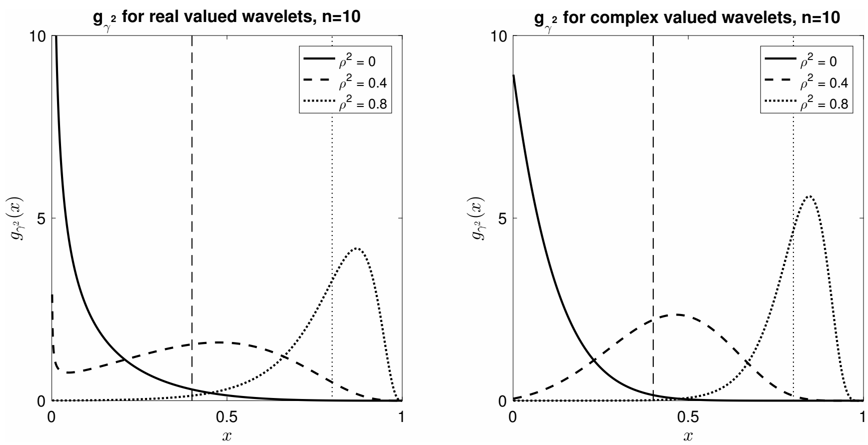

Figure 1: Asymptotic probability density functions for temporally smoothed wavelet coherence using real and complex valued wavelets. Degrees of freedom $n = 10$ and true coherence $\rho^2 = 0$, 0.4, and 0.8 (marked with the verticle lines)

It then follows that $\Omega^{(T)}(\tilde{a}, \tilde{b})$ is asymptotically $(1/n)\mathcal{W}_p\{n, S(f_{\tilde{a}})\}$ as $T \to \infty$ for all $(\tilde{a}, \tilde{b}) \in \tilde{\mathcal{T}}_{\alpha,\kappa}$, where $n = 1/\left(\sum_{l=1}^{\infty} \eta_l^2\right)$, again along a similar line of argument. The distribution to the wavelet coherence is slightly different for real valued wavelets. Let $\Omega^{(T)ij}(a, b)$ be the $\mathbb{R}^{2\times 2}$ matrix made up of the ith and jth columns and rows of $\Omega^{(T)}(a, b)$. It is immediate that asymptotically $\Omega^{(T)ij} \sim \mathcal{W}_2\{n, S(f_{\tilde{a}})\}$ and from Theorem 5.3.2 of Muirhead (1985) (first presented in Fisher (1928)) that $\gamma_{ij}^2(\tilde{a}, \tilde{b})$ asymptotically has density function

$$g_{\gamma^2}(x) = \frac{\Gamma(\frac{1}{2}n)}{\Gamma(\frac{1}{2})\Gamma\{\frac{1}{2}(n-1)\}} x^{-1/2}(1-x)^{(n-3)/2}(1-\rho^2)^{n/2}\ {}_2F_1(n/2, n/2, 1/2, \rho^2 x)$$

where ρ^2 is shorthand for $\rho_{ij}^2(f_{\tilde{a}})$, the spectral coherence between $N_i(t)$ and $N_j(t)$ at frequency $f_{\tilde{a}}$. In the case of $\rho_{ij}^2(f_{\tilde{a}}) = 0$, this distribution is Beta$\{1/2, (n-1)/2\}$. The density functions are shown in Figure 1.

The test of stationarity at scale j is also extensible to real valued wavelets. However, the test statistic becomes

$$V_j = K^{pKn/2} \frac{\prod_{i=1}^{K} \det\{\Omega^{(T)}(\tilde{a}_j, \tilde{b}_i)\}^{n/2}}{\det\left\{\sum_{i=1}^{K} \Omega^{(T)}(\tilde{a}_j, \tilde{b}_i)\right\}^{Kn/2}}.$$

The asymptotic distribution of V_j under H_j and the rate of convergence is identical to complex valued wavelets.

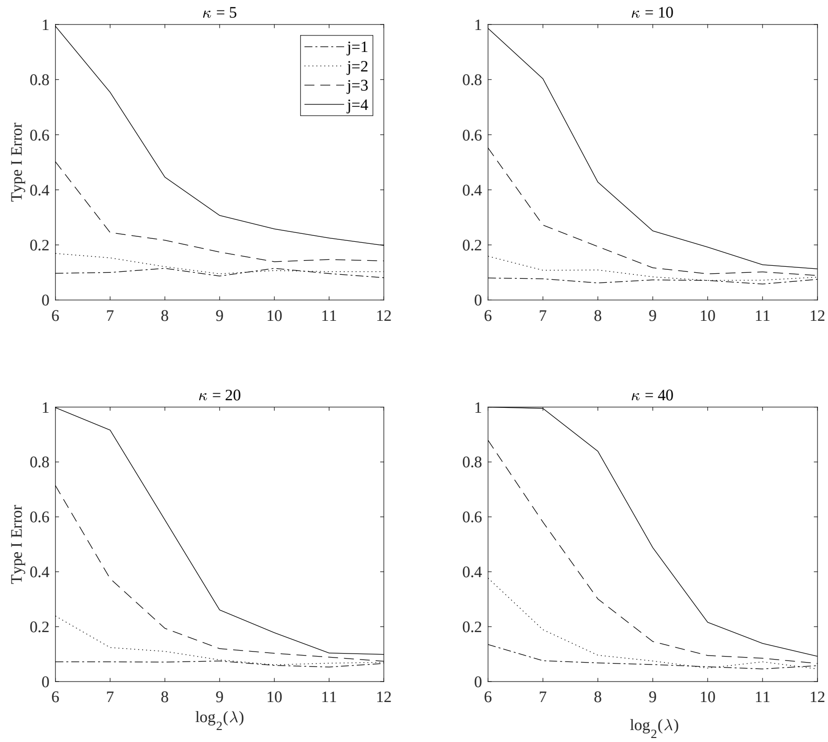

Figure 2: The type I error rate for testing H_1, H_2, H_3 and H_4 with a pair of independent homogeneous Poisson processes at $\kappa = 5, 10, 20$ and 40

3 Practical implementation

For the asymptotic distributions to be attained, growth rates of J and κ must be balanced such that $2^J\kappa = o(T^{1/2})$. To aid the simultaneous choice of κ and J in finite data settings, we perform analyses of the type I error rate of the stationarity test at the $\alpha = 0.05$ level. Since all point processes can undergo an arbitrary time rescaling, in this analysis all time scales are normalised to $(0, 1]$ and the intensity of events varied. For example, a homogeneous Poisson process with intensity λ observed on $(0, T)$ is equivalent to a homogeneous Poisson process with intensity $T\lambda$ observed on $(0, 1]$. We consider a pair of independent homogeneous Poisson processes, varying the intensity λ, for $\kappa = 5, 10, 20$ and 40 at scales $j = 1, 2, 3$ and 4. Results are shown in Fig. 2. Empirically, we find that for a dataset with N events (within the studied range), ensuring $J + \log_2(\kappa) < 1 + \frac{1}{2}\log_2(N)$ offers some conservative guidance for selecting parameters so as to attain the nominal level.

4 Code

Code and data associated with this paper can be found in the github repository

https://github.com/AlexGibberd/pointWav.

References

Barndorff-Nielsen, O. E. & Cox, D. R. (1989). *Asymptotic Techniques for Use in Statistics*. Chapman and Hall.

Brillinger, D. R. (1972). The spectral analysis of stationary interval functions. *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Theory of Statistics* , 483–513.

Fisher, R. A. (1928). The general sampling distribution of the multiple correlation coefficient. *Proceedings of the Royal Society Series A* **121**.

Muirhead, R. J. (1985). *Aspects of Multivariate Statistical Theory*. John Wiley & Sons, Inc, 2nd ed.

Walden, A. T. (2000). A unified view of multitaper multivariate spectral estimation. *Biometrika* **87**, 767–788.