Supplementary Online Appendix

Quantile regression models and the crucial difference between individual-level effects and population-level effects

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Content of supplementary online appendix:

Appendix A: Contributions and UQR-type influences

This section aims to clarify the difference between a contribution on a quantile value and the influence on a quantile value. To do so, we use a simple simulation, with 10% treated and a uniform treatment strength of minus one unit across the distribution. Here, we concentrate on the fifth quantile value $Q_{0.05}$, which is 8.14. Table 1 shows that any observation with an outcome equal to or below the fifth percentile receives a RIF-value of -1.77. Otherwise, the RIF-value is 8.67. 3.2% of all non-treated but 20.9% of all treated are part of the lowest 5%. The density around the fifth quantile is about 0.0958.

1A 1: Descriptive statistics for the distribution of RIF-values for the 5th percentile of a simulation setting with 10% treated and a uniform -1 unit outcome difference between groups.

	N	av. RIF	RIF: $y \leq Q_{0.05}$	RIF: $y > Q_{0.05}$	$P(y \leq Q_{0.05})$	$P(y \leq Q_{0.05})$
All	20000	8.14	-1.77	8.67	0.050	0.950
Not treated	18000	8.33	-1.77	8.67	0.032	0.968
Treated	2000	6.48	-1.77	8.67	0.209	0.790

We know that the β-coefficient of the UQR is the difference of expected RIF-values between groups:

$$\beta_{\tau,T} = E\left[RIF(y, Q_\tau)|T = 1\right] - E\left[RIF(y, Q_\tau)|T = 0\right] \tag{1}$$

In this case, we thus can easily calculate the coefficient as

$$\beta_{\tau,T} = 6.48 - 8.33 = -1.85.$$

If the non-treated influenced the outcome distribution the same way as the treated, the fifth percentile value would be 1.85 units lower. Note that this is not the same as the contribution of the treatment on the observed fifth percentile value. As we argued in the paper, the contribution of a treatment on a quantile value is a product of the treatment's influence with the share of the treated:

$$\begin{aligned} Q_\tau &= E[RIF(y, Q_\tau)] \qquad (2) \\ &= E_T\big[E[RIF(y, Q_\tau|T)]\big] \qquad (3) \\ &= E\big[RIF(y, Q_\tau|T)\big] \cdot P(T) \qquad (4) \end{aligned}$$

In addition, note that the RIF-equation can be rearranged in the following way:

$$
\begin{aligned}
RIF(y, Q_\tau) &= Q_\tau + \frac{\tau - \mathbb{1}[y \leq Q_\tau]}{f_Y(Q_\tau)} && (5)\\
&= \frac{\mathbb{1}[y > Q_\tau]}{f_Y(Q_\tau)} + Q_\tau + \frac{\tau - 1}{f_Y(Q_\tau)} && (6)\\
&= \underbrace{\frac{1}{f_Y(Q_\tau)}}_{c_{1,\tau}} \cdot \mathbb{1}[y > Q_\tau] + \underbrace{Q_\tau - \frac{1}{f_Y(Q_\tau)} \cdot (1 - \tau)}_{c_{2,\tau}}. && (7)
\end{aligned}
$$

Taking expectations over equation 7, we can see that the only variable part of the equation is the proportion of a group to have outcomes above the unconditional quantile value:

$$
\begin{aligned}
E\left[RIF(y, Q_\tau) | T\right] &= E\left[c_{1,\tau} \cdot \mathbb{1}[y > Q_\tau] | T\right] + E[c_{2,\tau} | T] && (8)\\
&= c_{1,\tau} \cdot P(y > Q_\tau | T) + c_{2,\tau} && (9)
\end{aligned}
$$

We can thus rephrase the interpretation of treatments influence in terms of these conditional probabilities: If we changed the probability to have units below the fifth quantile value for all units from $P(y > Q_{0.05}) = 0.968$ to $P(y > Q_{0.05}) = 0.790$, we would expect the overall distribution to shift strongly downwards, resulting in a reduction of the fifth percentile with 1.85 units. The UQR influence therefore expresses an expected shift of the overall distribution given two counterfactual distributions: one in which we assign all units $P(y > Q_\tau | T = 0)$ and one for which we assign $P(y > Q_\tau | T = 1)$ for all units. UQR implicitly calculates percentile values for both counterfactual distributions by holding everything else constant and changing only $P(y > Q_\tau)$ across groups. In this case, we can thus easily calculate these quantities using the information in table 1 and the density around the observed fifth quantile.

As we have shown in equation 4, the observed quantile is the sum of weighted, expected RIF-values across groups.

$$
Q_{0.05} = 6.48 \cdot 0.1 + 8.33 \cdot 0.9 = 8.14
$$

We can thus calculate the two counterfactuals $Q_{0.05}^{C_1}$ and $Q_{0.05}^{C_2}$ discussed above by substituting the expected RIF-values across groups:

$$
\begin{aligned}
Q_{0.05}^{C_1} &= E[\overbrace{RIF(y, Q_{0.05} | T = 1)}^{\text{RIF for T = 1}}] \cdot P(T = 1) + E[\overbrace{RIF(y, Q_{0.05} | T = 1)}^{\text{Substituted RIF for T = 1}}] \cdot P(T = 0)\\
&= 6.48 \cdot 0.1 + 6.48 \cdot 0.9 = 6.48
\end{aligned}
$$

$$
\begin{aligned}
Q_{0.05}^{C_2} &= E[\overbrace{RIF(y, Q_{0.05} | T = 0)}^{\text{Substituted RIF for T = 0}}] \cdot P(T = 1) + E[\overbrace{RIF(y, Q_{0.05} | T = 0)}^{\text{RIF for T = 0}}] \cdot P(T = 0)\\
&= 8.33 \cdot 0.1 + 8.33 \cdot 0.9 = 8.33
\end{aligned}
$$

The difference between $Q^{C_1}_{0.05}$ and $Q^{C_2}_{0.05}$ is -1.85, because the β-coefficient of the UQR expresses exactly the difference between these two counterfactual states of the unconditional distribution.

We can understand the nature of these counterfactuals further by substituting the expected RIF-value with weighted conditional probabilities as shown in equation 9.[1]

$$\begin{aligned}
Q^{C_1}_{0.05} &= \frac{1}{f_Y(Q_{0.05})} \cdot P(y > Q_{0.05}|T=1) \cdot P(T=1) \\
&+ \frac{1}{f_Y(Q_{0.05})} \cdot \overbrace{P(y > Q_{0.05}|T=1)}^{\text{Substituted probability}} \cdot P(T=0) \\
&= \left[\frac{1}{0.0958359} \cdot 0.791\right] \cdot 0.1 \\
&+ \left[\frac{1}{0.0958359} \cdot 0.791\right] \cdot 0.9 + c_{2,\tau} \\
&\approx 6.48.
\end{aligned}$$

$$\begin{aligned}
Q^{C_2}_{0.05} &= \frac{1}{f_Y(Q_{0.05})} \cdot \overbrace{P(y > Q_{0.05}|T=0)}^{\text{Substituted probability}} \cdot P(T=1) \\
&+ \frac{1}{f_Y(Q_{0.05})} \cdot P(y > Q_{0.05}|T=0) \cdot P(T=0) \\
&= \left[\frac{1}{0.0958359} \cdot 0.968\right] \cdot 0.1 \\
&+ \left[\frac{1}{0.0958359} \cdot 0.968\right] \cdot 0.9 + c_{2,\tau} \\
&\approx 8.33.
\end{aligned}$$

Again, the β-coefficient is exactly the difference between both counterfactuals. For the first, we hold everything else constant and set the conditional probability to have units above the *observed* fifth quantile value equal to the value of the treatment group. For the second, we repeat the exercise and assign the treated the conditional probability of the non-treated. Both would result in shifts of the overall distribution and the quantile values respectively.

[1] For brevity we abbreviate $Q_\tau - \frac{1}{f_Y(Q_\tau)} \cdot (1-\tau)$ to $c_{2,\tau}$ as in equation 7. This quantity is $8.14 - \frac{1}{0.0958359} \cdot 0.95 = -1.772778$ in this case.

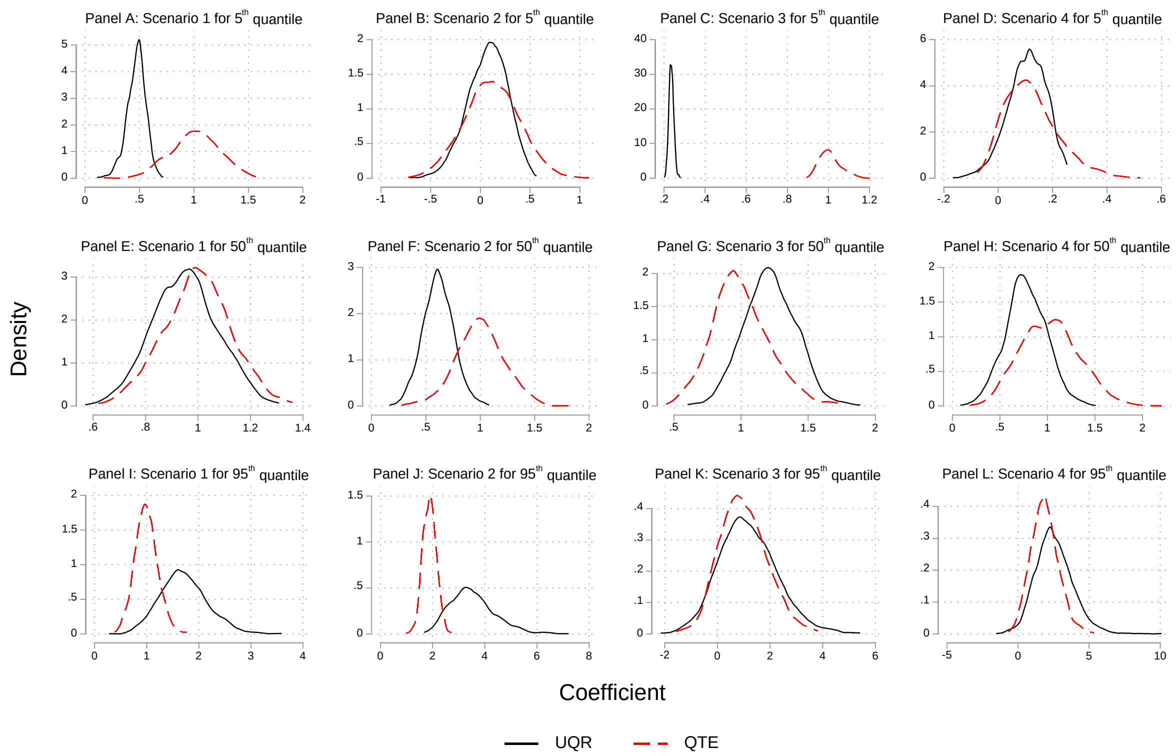
Online Appendix B: Supplementary tables and figures

Appendix Table B1: The means and the standard deviation of the simulations that are reported in Figure 3 (1,000 draws of N=1,000).

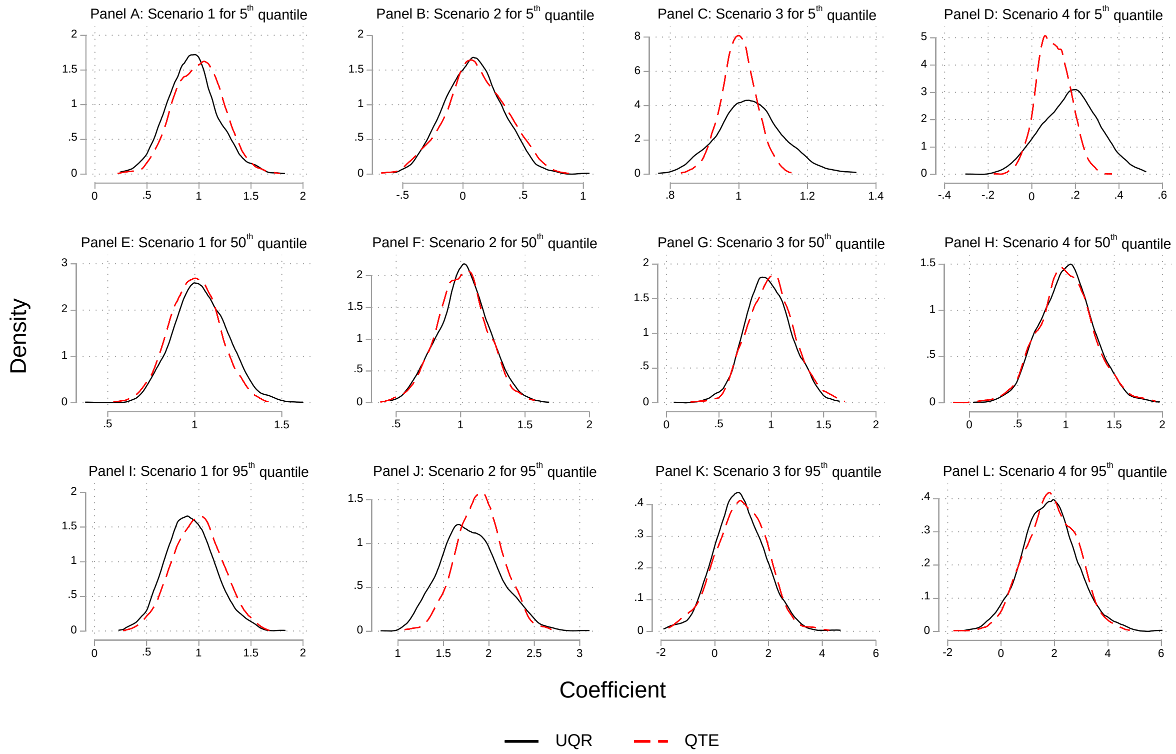
			Q^{10}				Q^{50}				Q^{90}			
	OLS		UQR		QTE		UQR		QTE		UQR		QTE	
Panel A: Scenario 1														
Binary treatment (90%=1)	0.998	(0.103)	1.552	(0.323)	0.989	(0.179)	0.945	(0.130)	1.005	(0.128)	0.554	(0.087)	1.004	(0.175)
Binary treatment (75%=1)	1.002	(0.073)	1.307	(0.203)	0.998	(0.123)	1.039	(0.107)	0.997	(0.094)	0.658	(0.087)	1.001	(0.123)
Binary treatment (50%=1)	0.999	(0.061)	0.915	(0.117)	1.002	(0.108)	1.118	(0.112)	1.002	(0.079)	0.912	(0.120)	1.006	(0.103)
Binary treatment (25%=1)	1.000	(0.076)	0.660	(0.087)	1.005	(0.122)	1.041	(0.108)	0.999	(0.091)	1.290	(0.209)	1.002	(0.126)
Binary treatment (10%=1)	0.997	(0.106)	0.549	(0.085)	1.012	(0.171)	0.950	(0.126)	0.997	(0.130)	1.537	(0.320)	0.987	(0.178)
Continuous treatment	1.003	(0.110)	0.980	(0.203)	0.992	(0.184)	1.035	(0.153)	0.999	(0.141)	0.983	(0.197)	0.994	(0.187)
Categorical treatment	1.001	(0.029)	0.798	(0.075)	0.999	(0.048)	1.254	(0.080)	0.999	(0.036)	0.797	(0.072)	1.001	(0.049)
Skewed treatment	1.030	(0.641)	0.977	(1.014)	0.981	(1.052)	0.999	(0.832)	0.955	(0.787)	0.992	(1.166)	0.971	(1.143)
Panel B: Scenario 2														
Binary treatment (90%=1)	1.006	(0.112)	0.277	(0.269)	0.196	(0.183)	1.500	(0.197)	0.997	(0.136)	0.892	(0.061)	1.803	(0.187)
Binary treatment (75%=1)	0.997	(0.085)	0.239	(0.160)	0.192	(0.134)	1.322	(0.166)	0.993	(0.113)	1.148	(0.084)	1.794	(0.136)
Binary treatment (50%=1)	0.999	(0.084)	0.184	(0.117)	0.205	(0.126)	0.981	(0.120)	1.005	(0.110)	1.880	(0.183)	1.798	(0.126)
Binary treatment (25%=1)	1.001	(0.104)	0.159	(0.121)	0.204	(0.158)	0.730	(0.110)	0.998	(0.138)	2.594	(0.393)	1.800	(0.155)
Binary treatment (10%=1)	1.009	(0.159)	0.137	(0.159)	0.213	(0.244)	0.613	(0.141)	1.000	(0.223)	2.274	(0.444)	1.781	(0.238)
Continuous treatment	1.000	(0.143)	0.191	(0.210)	0.209	(0.215)	1.022	(0.195)	1.009	(0.191)	1.821	(0.276)	1.804	(0.226)
Categorical treatment	1.004	(0.056)	0.189	(0.064)	0.199	(0.070)	1.076	(0.096)	0.998	(0.066)	1.703	(0.185)	1.798	(0.071)
Skewed treatment	1.011	(0.695)	0.208	(1.107)	0.193	(1.178)	0.971	(0.874)	0.997	(0.878)	1.849	(1.250)	1.808	(1.138)
Panel C: Scenario 3														
Binary treatment (90%=1)	0.998	(0.201)	1.353	(0.206)	0.990	(0.075)	0.799	(0.202)	0.975	(0.219)	0.799	(0.521)	0.954	(0.635)
Binary treatment (75%=1)	1.001	(0.145)	1.648	(0.192)	0.996	(0.052)	0.859	(0.149)	0.992	(0.146)	0.857	(0.401)	0.976	(0.439)
Binary treatment (50%=1)	1.008	(0.122)	1.060	(0.067)	1.000	(0.043)	0.969	(0.138)	1.002	(0.127)	0.954	(0.381)	1.019	(0.378)
Binary treatment (25%=1)	0.992	(0.143)	0.557	(0.027)	1.001	(0.050)	1.124	(0.150)	1.008	(0.146)	1.122	(0.509)	0.988	(0.449)
Binary treatment (10%=1)	1.006	(0.204)	0.408	(0.019)	1.006	(0.074)	1.216	(0.194)	0.999	(0.214)	1.155	(0.750)	0.968	(0.635)
Continuous treatment	0.989	(0.217)	1.267	(0.110)	0.999	(0.073)	0.972	(0.218)	1.001	(0.221)	0.967	(0.643)	0.992	(0.665)
Categorical treatment	1.004	(0.056)	0.912	(0.066)	1.000	(0.019)	1.133	(0.092)	1.001	(0.057)	0.914	(0.183)	1.009	(0.172)
Skewed treatment	1.065	(1.272)	1.530	(0.593)	1.029	(0.416)	0.986	(1.248)	0.978	(1.302)	0.872	(3.697)	1.020	(3.848)
Panel C: Scenario 4														

Binary treatment (90%=1)	1.002	(0.212)	0.424	(0.200)	0.195	(0.075)	1.151	(0.300)	0.988	(0.223)	1.374	(0.514)	1.812	(0.614)
Binary treatment (75%=1)	0.996	(0.154)	0.361	(0.117)	0.202	(0.060)	1.119	(0.207)	0.992	(0.160)	1.490	(0.391)	1.799	(0.461)
Binary treatment (50%=1)	0.995	(0.145)	0.272	(0.077)	0.201	(0.062)	1.007	(0.163)	1.002	(0.166)	1.753	(0.433)	1.797	(0.401)
Binary treatment (25%=1)	0.994	(0.179)	0.212	(0.066)	0.204	(0.081)	0.870	(0.168)	1.007	(0.199)	2.053	(0.583)	1.806	(0.476)
Binary treatment (10%=1)	1.000	(0.261)	0.177	(0.091)	0.221	(0.132)	0.774	(0.216)	1.026	(0.305)	2.219	(0.890)	1.792	(0.691)
Continuous treatment	0.997	(0.247)	0.289	(0.145)	0.199	(0.106)	1.010	(0.274)	1.000	(0.285)	1.739	(0.699)	1.792	(0.664)
Categorical treatment	1.002	(0.080)	0.273	(0.051)	0.202	(0.038)	1.026	(0.108)	1.004	(0.091)	1.724	(0.259)	1.801	(0.197)
Skewed treatment	1.059	(1.321)	0.293	(0.665)	0.233	(0.491)	1.022	(1.284)	1.085	(1.341)	1.835	(3.741)	1.985	(3.918)

Note: Standard deviations of the estimated coefficients in parentheses.



Appendix Figure B1: Distribution of the estimated treatment effect of the binary treatment variable with 10%=1 from the separate draws (1,000 draws of N=1,000). See Appendix Table B1 for the mean and standard deviation of the estimated treatment effects.



Appendix Figure B2: Distribution of the estimated treatment effect of the continuous treatment variable from the separate draws (1,000 draws of N=1,000). See Appendix Table B1 for the mean and standard deviation of the estimated treatment effects.

Online Appendix C: Supplementary empirical examples

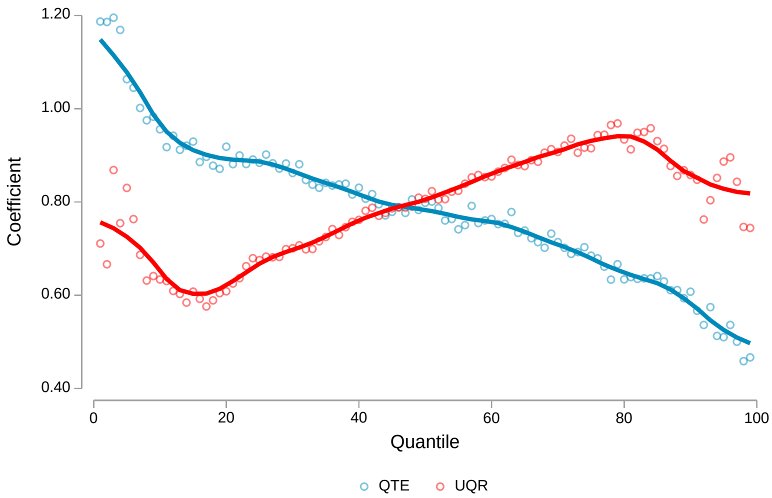
The main text's data simulations show how the proportion treated and the outcome distribution's shape impacts the differences between UQR and QTE coefficients. This online appendix illustrates the same points using real data.

Family Background and Academic Performance: The case of binary predictors

The first case study concerns the relation between family background and academic performance in the Panel Study of Income Dynamics (PSID) data, as studied in Grätz and Wiborg (2020). When estimating the association between parental education and academic achievements, the UQR and QTE models provide contradictory patterns across quantiles (Appendix Figure C1). The estimated QTE coefficients show a pattern where differences are largest at the bottom of the performance distribution and then monotonically decline across the distribution. These findings indicate that high-education parents focus resources on their children with the lowest academic performance, a compensatory behavior that may be motivated by parents' attempts to avoid children's downward mobility.

In contrast, UQR coefficients provide nearly the opposite pattern, mirroring our simulation results in the main text. These contradictory patterns are neither artifacts nor biases but rather caused by population-level influences being different from individual-level ones. Increasing the proportion of highly educated parents in the population would shift the whole achievement distribution upwards, since children of high-educated parents have higher achievement levels.[1] This shift would result in an unequal achievement distribution at the population level, magnifying the 90^{th}-10^{th} ratio, for example. This interpretation illustrates that the UQR model tells a different, although complementing, story compared to QTE models.

[1] More technically, the UQR coefficients tell us that the whole distribution would shift upwards if children of non-highly educated parents would influence the entire distribution in the same manner as children of highly educated parents. If we shift their distribution towards the highly educated, we push many low-achieving students upwards, resulting in larger increases of upper quantiles compared to lower ones.

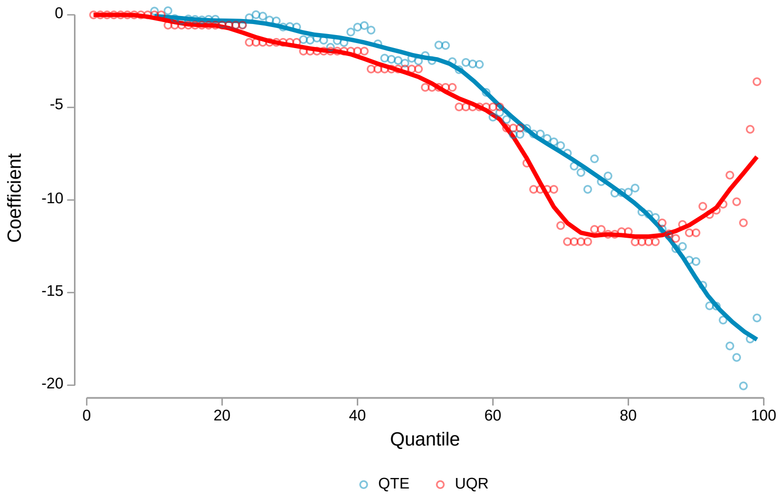


Appendix Figure C1: Associations between high parental education (16.6%=1) and academic achievements in PSID data (N=3569).

Note: Scatter points represents the estimated UQR (red dots) and QTE (blue dots) coefficients for quantiles 1 to 99 in steps of 1. The red and blue lines plots a local polynomial smooth of the respective coefficient on the quantile. Academic performance is measured through the Woodcock-Johnson rescaled test when respondents were aged 10–17. Parental education is measured as the highest level of education attained by either parent. For more details about data, see Grätz and Wiborg (2020). QTE coefficients are estimated using CQR (since no control variables are included), and UQR coefficients are estimated using RIF-OLS.

Scarring effects of unemployment: The case of skewed outcome variables

Trigger events such as unemployment may have scarring effects on workers' subsequent labor market outcomes (Gangl, 2006), including job tenure (Böheim & Taylor, 2002). Our second case study focuses on the influence of unemployment experience on tenure using data from the 2018 wave of the General Social Survey (GSS). This example is chosen because our simulations showed that UQR and QTE results could differ markedly for skewed distribution. The tenure variable in GSS is heavily right-skewed; while the median tenure is four years, the corresponding 75th and 95th percentiles are 11 and 28 years, respectively.



Appendix Figure C2: Unemployment effects on work duration with the firm in GSS data (N=1389).

Note: Scatter points represents the estimated UQR (red dots) and QTE (blue dots) coefficients for quantiles 1 to 99 in steps of 1. The red and blue lines plots a local polynomial smooth of the respective coefficient on the quantile. Unemployment is measured as whether workers had been unemployed within the past ten years (26%=1). The tenure variable is based on workers' self-reports on how long they have worked in their present job for their current employer. All models include controls for full-time work, unionization status, race, industry, years of education, occupational prestige, age, gender, married, and number of children. The QTE coefficients are estimated using the GQR model, while the UQR coefficients are estimated using RIF-OLS.

Appendix Figure C2 displays the UQR and QTE coefficients for quantiles 50 to 99. Differences between QTE coefficients and UQR coefficients are most pronounced at the top of the tenure distribution. The estimated individual-level unemployment scar effects worsen monotonically across the tenure distribution, from about two years at the median to approximately 18 years at the top. In contrast, the UQR estimates show a U-shaped pattern with the largest influences between the 70th and 90th percentiles.

Online Appendix D: Replication files

Data and code to replicate the results are available on SocArXiv using an anonymized view-only link. Readers can access Appendix D by using the following link:

https://osf.io/quv7a/?view_only=392566a330a847f4b48327fdd5d44943