Supporting Information

File S7. Fisher Information Calculation

The calculation of the Fisher information needed to estimate confidence intervals of a piece-wise constant trajectory of population sizes, requires the following expected values:

$$\mathrm{E}[z_j^i z_k^l \mid \mathbf{Y}] = \begin{cases} \mathrm{E}[z_j^i \mid \mathbf{Y}] & j = k, i = l \\ 0 & j \neq k, i = l \\ \mathrm{E}[z_j^i \mid \mathbf{Y}]\mathrm{E}[z_k^l \mid \mathbf{Y}] & i \neq l \end{cases}$$

$$\mathrm{E}[\Delta_j^i \Delta_k^l \mid \mathbf{Y}] = \begin{cases} \Delta_{j,k}^i & i = l \\ \mathrm{E}[\Delta_j^i \mid \mathbf{Y}]\mathrm{E}[\Delta_k^l \mid \mathbf{Y}] & i \neq l \end{cases}$$

$$\mathrm{E}[z_j^i \Delta_k^l \mid \mathbf{Y}] = \begin{cases} \mathrm{E}[z_j^i \mid \mathbf{Y}]\mathrm{E}[\Delta_k^l \mid \mathbf{Y}] & i \neq l \\ \mathrm{E}[z_j^i \Delta_j^i \mid \mathbf{Y}] & i = l, j = k \\ 0 & i = l, j < k \\ (z\Delta)_{jk}^i & k < j \end{cases}$$

For $k < j$ and $i \in \mathcal{I}$

$$(z\Delta)_{jk}^i = (x_{k+1}^i - x_k^i)\frac{\sum_{l=1}^{k-1} \hat{F}_{l,j}^i \hat{P}_{l,j}^i}{\sum_{j=1}^{D_i-1}\sum_{k=j}^{D_i-1} \hat{F}_{k,j}^i \hat{P}_{k,j}^i}$$

$$+ \int_{x_k^i}^{x_{k+1}^i} (x_{k+1}^i - u) \exp\left\{-\frac{(x_{k+1}^i - u)A^i(x_{k+1}^i)}{N(x_{k+1}^i)}\right\} du \frac{\hat{F}_{k,j}^i \frac{\hat{P}_{k,j}^i}{\hat{Q}_k^i}}{\sum_{j=1}^{D_i-1}\sum_{k=j}^{D_i-1} \hat{F}_{k,j}^i \hat{P}_{k,j}^i}$$

and for $k < j$ and $i \in \mathcal{I}^c$

$$(z\Delta)_{jk}^i = z_j^i E[\Delta_k^i \mid \mathbf{Y}]$$

For $j < k$ and $i \in \mathcal{I}$

$$\Delta_{j,k}^i = (x_{j+1}^i - x_j^i)(x_{k+1}^i - x_k^i)\frac{\sum_{l=1}^{j-1}\sum_{m=k+1}^{D_i-1} \hat{F}_{l,m}^i \hat{P}_{l,m}^i}{\sum_{j=1}^{D_i-1}\sum_{k=j}^{D_i-1} \hat{F}_{k,j}^i \hat{P}_{k,j}^i}$$

$$+(x_{k+1}^i - x_k^i) \int_{x_j^i}^{x_{j+1}^i} (x_{j+1}^i - u) \exp\left\{-\frac{(x_{j+1}^i - u)A^i(x_{j+1}^i)}{N(x_{j+1}^i)}\right\} du \frac{\sum_{l=k+1}^{D_i-1} \hat{F}_{j,l}^i \frac{\hat{P}_{j,l}^i}{\hat{Q}_j^i}}{\sum_{j=1}^{D_i-1}\sum_{k=j}^{D_i-1} \hat{F}_{k,j}^i \hat{P}_{k,j}^i}$$

$$+ \int_{x_j^i}^{x_{j+1}^i}\int_{x_k^i}^{x_{k+1}^i} (x_{j+1}^i - u)(t - x_k^i) \exp\left\{-\frac{(x_{j+1}^i - u)A^i(x_{j+1}^i)}{N(x_{j+1}^i)} - \frac{(t - x_k^i)A^i(x_{k+1}^i)}{N(x_{k+1}^i)}\right\} dudt \frac{\hat{F}_{j,k}^i \frac{\hat{P}_{j,k}^i}{\hat{Q}_j^i (1-q_k^i)}}{\sum_{j=1}^{D_i-1}\sum_{k=j}^{D_i-1} \hat{F}_{k,j}^i \hat{P}_{k,j}^i}$$

and

$$\Delta_{j,j}^i = (x_{j+1}^i - x_j^i)^2 \frac{\sum_{k=1}^{j-1}\sum_{l=j+1}^{D_i-1} \hat{F}_{k,l}^i \hat{P}_{k,l}^i}{\sum_{j=1}^{D_i-1}\sum_{k=j}^{D_i-1} \hat{F}_{k,j}^i \hat{P}_{k,j}^i}$$

$$+\int_{x_j^i}^{x_{j+1}^i}(x_{j+1}^i-u)^2\exp\left\{-\frac{(x_{j+1}^i-u)A^i(x_{j+1}^i)}{N(x_{j+1}^i)}\right\}du\frac{\sum_{k=j+1}^{D_i-1}\hat{F}_{j,k}^i\frac{\hat{P}_{j,k}^i}{\hat{Q}_j^i}}{\sum_{j=1}^{D_i-1}\sum_{k=j}^{D_i-1}\hat{F}_{k,j}^i\hat{P}_{k,j}^i}$$

$$+\int_{x_j^i}^{x_{j+1}^i}\int_u^{x_{j+1}^i}(t-u)^2\frac{1}{N(x_{j+1}^i)}\exp\left\{-\frac{(t-u)A^i(x_{j+1}^i)}{N(x_{j+1}^i)}\right\}dtdu\frac{\hat{F}_{j,j}}{\sum_{j=1}^{D_i-1}\sum_{k=j}^{D_i-1}\hat{F}_{k,j}^i\hat{P}_{k,j}^i}$$

$$+\int_{x_j^i}^{x_{j+1}^i}(t-x_j^i)^2\exp\left\{-\frac{(t-x_j^i)A^i(x_{j+1}^i)}{N(x_{j+1}^i)}\right\}dt\frac{\sum_{k=1}^{j-1}\hat{F}_{k,j}^i\frac{\hat{P}_{k,j}^i}{1-\hat{q}_j^i}}{\sum_{j=1}^{D_i-1}\sum_{k=j}^{D_i-1}\hat{F}_{k,j}^i\hat{P}_{k,j}^i}$$

For $i\in\mathcal{I}^c$ and $j<k$

$$\Delta_{j,k}^i=\begin{cases}0 & \sum_{l=1}^{j}I^i(x_{l+1}^i)=0 \text{ or } y_j^i=0\\ (x_{j+1}^i-x_j^i)(x_{k+1}^i-x_k^i) & I^i(x_{j+1}^i)=0,\sum_{l=1}^{j}I^i(x_{l+1}^i)>0,y_j^i=1,y_k^i=1\\ (x_{k+1}^i-x_k^i)\delta_j^i & I^i(x_{j+1}^i)=1,\sum_{l=1}^{j}I^i(x_{l+1}^i)>0,y_k^i=1\end{cases},$$

where δ_j^i is as defined in Equation 29, and

$$\Delta_{j,j}^i=\begin{cases}0 & \sum_{l=1}^{j}I^i(x_{l+1}^i)=0 \text{ or } y_j^i=0\\ (x_{j+1}^i-x_j^i)^2 & I^i(x_{j+1}^i)=0,\sum_{l=1}^{j}I^i(x_{l+1}^i)>0,y_j^i=1\\ \delta_{j,j}^i & I^i(x_{j+1}^i)=1,\sum_{l=1}^{j}I^i(x_{l+1}^i)>0,y_j^i>0\end{cases},$$

where

$$\delta_{j,j}^i=(x_{j+1}^i-x_j^i)^2\left[\frac{\sum_{k=1}^{j-1}I^i(x_{k+1}^i)\hat{Q}_k^i\prod_{l=k+1}^{D_i-1}[\hat{q}_l^i]^{y_l^i}}{\sum_{k=1}^{D_i-1}I^i(x_{k+1}^i)\hat{Q}_k^i\prod_{l=k+1}^{D_i-1}[\hat{q}_l^i]^{y_l^i}}\right]$$

$$+\int_{x_j^i}^{x_{j+1}^i}(x_{j+1}^i-u)^2\exp\left\{-\frac{(x_{j+1}^i-u)A^i(x_{j+1})}{N(x_{j+1}^i)}\right\}du\left[\frac{I^i(x_{j+1}^i)\prod_{l=j+1}^{D_i-1}[\hat{q}_l^i]^{y_l^i}}{\sum_{k=1}^{D_i-1}I^i(x_{k+1}^i)\hat{Q}_k^i\prod_{l=k+1}^{D_i-1}[\hat{q}_l^i]^{y_l^i}}\right]$$

and

For $i\in\mathcal{I}$

$$\mathrm{E}[z_j^i\Delta_j^i\mid\mathbf{Y}]=\int_{x_j^i}^{x_{j+1}^i}\int_u^{x_{j+1}^i}(t-u)\frac{1}{N(x_{j+1}^i)}\exp\left\{-\frac{(t-u)A^i(x_{j+1}^i)}{N(x_{j+1}^i)}\right\}dtdu\frac{\hat{F}_{j,j}}{\sum_{j=1}^{D_i-1}\sum_{k=j}^{D_i-1}\hat{F}_{k,j}^i\hat{P}_{k,j}^i}$$

$$+\int_{x_j^i}^{x_{j+1}^i}(t-x_j^i)\exp\left\{-\frac{(t-x_j^i)A^i(x_{j+1}^i)}{N(x_{j+1}^i)}\right\}dt\frac{\sum_{k=1}^{j-1}\hat{F}_{k,j}^i\frac{\hat{P}_{k,j}^i}{1-\hat{q}_j^i}}{\sum_{j=1}^{D_i-1}\sum_{k=j}^{D_i-1}\hat{F}_{k,j}^i\hat{P}_{k,j}^i}.$$

and for $i\in\mathcal{I}^c$

$$\mathrm{E}[z_j^i\Delta_j^i\mid\mathbf{Y}]=\delta_j^i z_j^i$$

The gradient vector of the complete data log-likelihood has lth element

$$\frac{\partial}{\partial\log N_l}\mathcal{L}_c(\mathbf{Y}_c;\hat{\mathbf{N}})=B_l-A_l+C_l-Z_l \tag{1}$$

With

$$A_l=\sum_{j=1}^{D}a_j^0 1_{l,j}^0$$

$$B_l = \sum_{j=1}^{D} 0.5 A^0(x^0_{j+1})[A^0(x^0_{j+1}) - 1](x^0_{j+1} - x^0_j) 1^0_{l,j} \exp[-\log N_l],$$

$$C_l = \sum_{i=0}^{m-2} \sum_{j=1}^{D} A^i(x^i_{j+1}) \Delta^i_j 1^i_{l,j} \exp[-\log N_l],$$

and

$$Z_l = \sum_{i=0}^{m-2} \sum_{j=1}^{D} z^i_j 1^i_{l,j}.$$

Next, differentiating Equation 1 in this file, we have $\frac{\partial^2 l_c(\mathbf{Y}_c;\hat{\mathbf{N}})}{\partial \log N_l \partial \log N_m} = 0$ for all $l \neq m$, so the Hessian is a diagonal matrix with (l,l)th element

$$\frac{\partial^2}{\partial \log N_l^2} \mathcal{L}_c(\mathbf{Y}_c; \hat{\mathbf{N}}) = -B_l - C_l$$

and

$$\mathrm{E}\left[\left(\frac{\partial l_c(\mathbf{Y}_c;\hat{\mathbf{N}})}{N_l}\right)^2 \mid \mathbf{Y}\right] = (B_l - A_l)^2 + 2(B_l - A_l)\mathrm{E}[C_l - Z_l \mid \mathbf{Y}] + \mathrm{E}[(C_l - Z_l)^2 \mid \mathbf{Y}]$$

where

$$\mathrm{E}[C_l^2 \mid \mathbf{Y}] = \exp[-2logN_l] \sum_{i=0}^{m-2} \sum_{j=1}^{D_i - 1} \left[\left\{A^i(x^i_{j+1})\right\}^2 \Delta^i_{j,j} 1^i_{l,j} + 2 \sum_{k=j+1}^{D_i-1} A^i(x^i_{j+1}) A^i(x^i_{k+1}) \Delta^i_{j,k} 1^i_{l,j} 1^i_{l,k} \right]$$

$$+2\exp[-2logN_l] \sum_{i=0}^{m-2} \sum_{j=1}^{D_i-1} \left[A^i(x^i_{j+1}) \mathrm{E}[\Delta^i_j \mid \mathbf{Y}] 1^i_{l,j} \sum_{p=i+1}^{m-2} \sum_{k=1}^{D_p-1} A^p(x^p_{k+1}) \mathrm{E}[\Delta^p_k \mid \mathbf{Y}] 1^p_{l,k} \right],$$

$$\mathrm{E}[Z_l^2 \mid \mathbf{Y}] = \sum_{i=0}^{m-2} \sum_{j=1}^{D_i-1} \left[\mathrm{E}[z^i_j \mid \mathbf{Y}] 1^i_{l,j} + 2\mathrm{E}[z^i_j \mid \mathbf{Y}] 1^i_{l,j} \sum_{p=i+1}^{m-2} \sum_{k=1}^{D_p-1} \mathrm{E}[z^p_k \mid \mathbf{Y}] 1^p_{l,k} \right]$$

and

$$\mathrm{E}[C_l Z_l \mid \mathbf{Y}] = \frac{1}{N_l} \sum_{i=0}^{m-2} \sum_{j=1}^{D} A^i(x^i_{j+1}) \mathrm{E}[z^i_j \Delta^i_j \mid \mathbf{Y}] 1^i_{l,j} + \sum_{i=0}^{m-2} \sum_{j=1}^{D} A^i(x^i_{j+1}) \sum_{k=j+1}^{D} (z\Delta)^i_{k,j} 1^i_{l,k} 1^i_{l,j}$$

$$+ \sum_{i=0}^{m-2} \sum_{j=1}^{D} A^i(x^i_{j+1}) \sum_{p=1, p \neq i}^{m-2} \sum_{k=1}^{D_p - 1} \mathrm{E}[\Delta^i_j \mid \mathbf{Y}] \mathrm{E}[z^p_k \mid \mathbf{Y}] 1^i_{l,j} 1^p_{l,k}$$

Also,

$$\mathrm{E}\left[\left(\frac{\partial l_c(\mathbf{Y}_c;\hat{\mathbf{N}})}{N_l}\right)\left(\frac{\partial l_c(\mathbf{Y}_c;\hat{\mathbf{N}})}{N_k}\right) \mid \mathbf{Y}\right] = (B_l - A_l)(B_k - A_k) + (B_l - A_l)\mathrm{E}[C_k - Z_k \mid \mathbf{Y}]$$

$$+(B_k - A_k)\mathrm{E}[C_l - Z_l \mid \mathbf{Y}] + \mathrm{E}[(C_l - Z_l)(C_k - Z_k) \mid \mathbf{Y}]$$

where

$$\mathrm{E}[C_l C_k] = \exp[-logN_l - logN_k] \sum_{i=0}^{m-2} \sum_{j=1}^{D_i-1} A^i(x^i_{j+1}) 1^i_{l,j} \sum_{o=1}^{m-2} \sum_{p=1}^{D_o-1} A^o(x^o_{p+1}) 1^o_{k,p} \mathrm{E}[\Delta^i_j \Delta^o_p \mid \mathbf{Y}],$$

$$\mathrm{E}[Z_l Z_k] = \sum_{i=0}^{m-2} \sum_{j=1}^{D_i-1} \sum_{o \neq i}^{m-2} \sum_{p=1}^{D_o-1} 1^i_{l,j} 1^o_{k,p} \mathrm{E}[z^i_j z^o_p \mid \mathbf{Y}]$$

and for $l < o$

$$\mathrm{E}[C_l Z_o \mid \mathbf{Y}] = \frac{1}{N_l} \sum_{i=0}^{m-2} \sum_{j=1}^{D} A^i(x^i_{j+1}) 1^i_{l,j} \left\{ \sum_{k=j+1}^{D} (z\Delta)^i_{k,j} 1^i_{o,k} + \sum_{p=1, p\neq i}^{m-2} \sum_{k=1}^{D} \mathrm{E}[\Delta^i_j \mid \mathbf{Y}] \mathrm{E}[z^p_k \mid \mathbf{Y}] 1^p_{o,k} \right\}$$