Bayesian joint modeling of the health profile and demand of home care patients

RAFFAELE ARGIENTO
Consiglio Nazionale delle Ricerche (CNR), Istituto di Matematica Applicata e Tecnologie Informatiche (IMATI), Milan, Italy

ALESSANDRA GUGLIELMI
Politecnico di Milano, Dipartimento di Matematica, Milan, Italy

ETTORE LANZARONE*
Consiglio Nazionale delle Ricerche (CNR), Istituto di Matematica Applicata e Tecnologie Informatiche (IMATI), Milan, Italy
*Corresponding author: Email: ettore.lanzarone@cnr.it

INAD NAWAJAH
Hebron University, Department of Mathematics, Hebron, Palestine

SUPPLEMENTARY MATERIAL

A. Expressions of the full-conditional distributions

Though we have used JAGS to simulate from the posterior distribution of the parameters, which automatically computes full conditional distributions, we sketch those distributions here for those readers interested in using this model by designing their own code.

We start from (2.7), with \( \theta_N = (\gamma_1, \gamma_2, \mu, \sigma^2, \lambda_1, \ldots, \lambda_R) \) and \( \theta_{CP} = (\beta_2, \beta_{1,1}, \ldots, \beta_{1,R}, \sigma_{\beta_1}, z_1, \ldots, z_R) \). Thanks to (2.7), the two blocks of parameters are posteriori independent, so that:

\[
\pi(\theta_N|\theta_{CP}, \text{data}) \propto \text{Lik}(\theta_N) \pi(\theta_N)
\]

\[
\pi(\theta_{CP}|\theta_N, \text{data}) \propto \text{Lik}(\theta_{CP}) \pi(\theta_{CP}).
\]

Before listing all of the full conditionals, we mention that, when they are non-standard distributions, either an acceptance-rejection algorithm or a Metropolis-Hastings step is needed for the actual implementation of the Gibbs sampler. We denote by \( \pi(\cdot|\text{all}) \) the distribution of the parameter \( \cdot \) conditionally to the rest of the parameters, to data and to covariates.

A.1 Full conditional of \( \gamma \) in \( \theta_N \)

Since \( \gamma = (\gamma_1, \gamma_2) \), its full conditional is a distribution proportional to the factor in the likelihood containing \( \gamma \) itself times its marginal prior:

\[
\pi(\gamma|\text{all}) \propto \prod_{t=1}^n \left( \prod_{i \in \Omega(t)} \text{Pois}(n_i; \lambda_{CP_t} e^{\gamma_1}, \gamma_2) \right) N \left( \gamma_1; f, \sigma^2_f \right) N \left( \gamma_2; f, \sigma^2_f \right).
\]

© The authors 2016. Published by Oxford University Press on behalf of the Institute of Mathematics and its Applications. All rights reserved.
where \( \text{Pois}(n_t; \lambda) \) represents the density of a Poisson random variable with parameter \( \lambda \) evaluated at the observed number of nurse visits \( n_t \), and \( N(*) \) denotes the density of a Gaussian random variable with parameters \( \mu \) and \( \sigma^2 \), evaluated at *. Hence:

\[
\pi(\gamma|\text{all}) \propto \prod_{i=1}^{n} \left( \prod_{t=1}^{T_{(i)}} e^{-\lambda_{(i)}} e^{x_{(i)t}'} \right) \sum_{i=1}^{n} \left( \prod_{t=1}^{T_{(i)}} e^{x_{(i)t}'} \right) \sum_{d=1}^{D} \sum_{r=1}^{R} \prod_{i=1}^{n} \left( \prod_{t=1}^{T_{(i)}} e^{x_{(i)t}'} \right) \sum_{d=1}^{D} \sum_{r=1}^{R} \prod_{i=1}^{n} \left( \prod_{t=1}^{T_{(i)}} e^{x_{(i)t}'} \right)
\]

This is a non-standard density.

A.2 Full conditional of \( \lambda_r \) in \( \theta_N \)

Similarly to the full conditional of \( \gamma \), we have:

\[
\pi(\lambda_r|\text{all}) \propto e^{-\lambda_r \sum_{i,c_{i,c}} e^{\mu_{i,c}'} \sum_{i,c_{i,c}} e^{\mu_{i,c}'} \log N(\lambda_r; \mu_\lambda, \sigma_\lambda^2)} \quad \forall r = 1, \ldots, R
\]

where \( \log N(*) \) denotes the log-normal density with parameters \( \mu \) and \( \sigma^2 \). This is again a non-standard density. However, it is worth mentioning that it could be more convenient to choose a gamma density as a prior for the random effects parameters \( \lambda_r \), so that the corresponding full conditional would be a gamma density again; in this case, the Metropolis-Hastings step can be avoided.

A.3 Full conditional of \( \mu_\lambda, \sigma_\lambda^2 \) in \( \theta_N \)

These parameters belong to the last level in the hierarchy of the model, so that their full conditional only depends on \( \lambda_1, \ldots, \lambda_R \). It is straightforward to check that the full conditional coincides with the posterior of this Bayesian model:

\[
\log(\lambda_1), \ldots, \log(\lambda_R) | \mu_\lambda, \sigma_\lambda^2 \text{ iid } \sim N(\mu_\lambda, \sigma_\lambda^2)
\]

\[
\mu_\lambda \sim N(\mu_0, \sigma_{\mu_\lambda}^2), \quad \sigma_\lambda \sim U(0, \sigma_{\sigma_\lambda}^2), \quad \mu_\lambda \text{ and } \sigma_\lambda \text{ independent}.
\]

Therefore, the full conditional of \( \mu_\lambda \) is:

\[
\mu_\lambda|\text{all} \sim N(a,b); \quad b = \left( \frac{R}{\sigma_{\mu_\lambda}^2} + \frac{1}{\sigma_{\mu_\lambda}^2} \right); \quad a = b^{-1} \left( \frac{\sum_{r=1}^{R} \log(\lambda_r)}{\sigma_{\mu_\lambda}^2} + \frac{\mu_0}{\sigma_{\mu_\lambda}^2} \right).
\]

As far as \( \sigma_\lambda^2 \) is concerned, observe first that the uniform prior of \( \sigma_\lambda \) is equivalent to the following prior for \( \sigma_\lambda^2 \):

\[
\pi(\sigma_\lambda^2) = \frac{1}{\sigma_{\sigma_\lambda}^2} \frac{1}{2\sqrt{\sigma_{\sigma_\lambda}^2}} |_{0, \sigma_{\sigma_\lambda}^2}(\sigma_\lambda^2)
\]
where $I_A(x) = 1$ if $x \in A$ and 0 otherwise. Hence, the full conditional of $\sigma_A^2$ is as follows:

$$
\pi(\sigma_A^2 | \text{all}) \propto \prod_{r=1}^{R} \frac{1}{\sqrt{\sigma_A^2}} \exp \left\{ -\frac{1}{2} \left( \frac{\mu_A - \log(\lambda_A)}{\sigma_A^2} \right)^2 \right\} \frac{1}{\sqrt{\sigma_A^2}} I_0(0.\sigma_A^2)(\sigma_A^2)
$$

$$
\propto \frac{1}{\sigma_A^2} \exp \left\{ -\frac{1}{2} \sum_{r=1}^{R} \left( \frac{\mu_A - \log(\lambda_A)}{2} \right)^2 \right\} I_0(0.\sigma_A^2)(\sigma_A^2)
$$

$$
= \text{inv} - \text{gamma} \left( \sigma_A^2 ; R + \frac{1}{2} + 1, \frac{1}{2} \sum_{r=1}^{R} (\mu_A - \log(\lambda_A))^2 \right) I_0(0.\sigma_A^2)(\sigma_A^2).
$$

A.4 Full conditional of $P$ in $\theta_{CP}$

We examine now the full conditionals of all parameters in $\theta_{CP}$; in particular we start with the transition matrix $P$, i.e., we consider the full conditionals of its first $R$ rows $(P_{r,1}, \cdots, P_{r,R+1})$, $r = 1 \ldots, R$.

Looking at the law of the CP process as expressed in (2.3), the posterior of the parameter $P$ only depends on the observed transitions. For each $r \in \{1, \cdots, R\}$ let us denote by $m_{ri}, l \in \{1, \cdots, R+1\}$ the number of transitions from state $r$ to state $l$ in our sample; clearly, $m_{rr} = 0$ for $r = 1, \cdots, R$. Considering that $\mathcal{Z}(\eta_{ij}|\eta_{i,j-1}) = P_{\eta_{i,j-1}j'}$, we have that for each $r = 1, \cdots, R$:

$$
\pi(P_{r,1}, \cdots, P_{r,R+1}|\text{all}) \prod_{l \in \{1, \cdots, R+1\} \setminus r} P_{l,r}^{m_{rl}} \pi(P_{r,1}, \cdots, P_{r,R+1}).
$$

This means that the full conditional $\pi(P_{r,1}, \cdots, P_{r,R+1}|\text{all})$ is a Dirichlet density with parameters $(a_1 + n_{r,1}, a_2 + n_{r,2}, \cdots, a_{R+1} + n_{r,R+1})$.

A.5 Full conditional of $z_r$ in $\theta_{CP}$

Let us remind that the density of a negative binomial random variable with parameters $z > 0$ and $q \in (0, 1)$, evaluated at $h = 1, 2, \ldots$, is:

$$
\text{NB}(h; z, q) = \binom{h + z - 2}{h - 1} q^{h-1}(1-q)^z, \quad \text{where} \quad \frac{h + z - 2}{h - 1} = \frac{\Gamma(h + z - 1)}{\Gamma(h)\Gamma(z)}.
$$

For each $r = 1, \ldots, R$, we have:

$$
\pi(z_r|\text{all}) \propto \prod_{i:j_{1i}=r} \Gamma(h_{ii} + z_r - 1) \sum_{j > 2, i:j_{ij}=r} \frac{\Gamma(h_{ij} + z_r - 1)}{\Gamma(z_r)} (1 - q_{ij}^r) \text{gamma}(z_r; a_0, a_0)
$$

$$
\propto \prod_{i:j_i=r} \Gamma(h_{ii} + z_r - 1) \sum_{j > 2, i:j_{ij}=r} \frac{\Gamma(h_{ij} + z_r - 1)}{\Gamma(z_r)} (1 - q_{ij}^r) \text{gamma}(z_r; a_0, a_0),
$$

where $\text{gamma}(z_r; a_0, a_0)$ is the gamma density with parameters $(a_0, a_0)$ evaluated at $z_r$.

A.6 Full conditional of $\beta_{1,r}$ in $\theta_{CP}$

Similarly to the full conditional of $z_r$, we have that, for each $r = 1, \cdots, R$:

$$
\pi(\beta_{1,r}|\text{all}) \propto \prod_{i:j_i=r} \frac{h_{ij}^{-1}}{(1-q_{ij}^r)} N(\beta_{1,r}; \beta_{01}, \sigma_{\beta_1}^2).
$$
where quantities $q_{ij}$ depend on $\beta_1, r$ according to the logit functions in (2.5).

A.7  Full conditional of $\beta_2$ in $\theta_{CP}$

The derivation is very similar to that derived at point A.6; however, $\beta_2$ does not appear in the distribution of the first holding time. Thus, we have:

$$
\pi(\beta_2 | \text{all}) \propto \prod_{i=1}^{n} \prod_{j=2}^{J(i)} \left( q_{j,h_{ij}-1} (1 - q_{ij})^{z_{ij}} \right) N(\beta_2; f, \sigma_f^2).
$$

A.8  Full conditional of $\sigma_{\beta_1}$ in $\theta_{CP}$

Parameter $\sigma_{\beta_1}$ belongs to the last level of the hierarchy in the model, so that its full conditional only depend on $\beta_1, 1, \cdots, \beta_1, R$ and it is the posterior density of the following Bayesian model:

$$
\begin{align*}
\beta_1, 1, \cdots, \beta_1, R & \iid N(\beta_{01}, \sigma_{\beta_1}^2) \\
\sigma_{\beta_1} & \sim U(0, \sigma_{0\beta_1}).
\end{align*}
$$

Mimicking the calculations made at point A.3, we obtain the following expression as the full conditional of $\sigma_{\beta_1}^2$:

$$
\text{inv-gamma} \left( \frac{\sigma_{\beta_1}^2 R + 1}{2} + 1, \frac{1}{2} \sum_{r=1}^{R} (\beta_{01} - \beta_r)^2 \right) I_{(0, \sigma_{0\beta_1})}(\sigma_{\beta_1}^2).
$$