Supplementary material to "Portfolio management in a stochastic factor model under the existence of private information"

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This supplementary material is intended to supplement the paper and contains references to parts of it. It is divided into two sections. In Section A, we solve the proposed model for other utility functions, namely when the manager operates under logarithmic and power preferences. In Section B, we provide an extension of the proposed model in the case of multiple risky assets and multiple stochastic factors.

Keywords: Portfolio management, Stochastic factor model, Initial enlargement of filtrations, Hamilton-Jacobi-Bellman equation

A. Other utility functions

In order to demonstrate the effectiveness of our approach, we provide solutions, in feedback form, for the optimal investment strategy and the optimal value function, in the special cases of the logarithmic and power utility functions.

A.1 The case of the logarithmic utility function

We assume that the informed economic agent operates under logarithmic preferences, that is, a utility function of the form

$$U(x) = \log(x).$$

(A.1)

THEOREM A.1 Assume logarithmic preferences (equation (A.1)). Then, the value function for the informed economic agent which is also the solution of the stochastic optimal control problem (3.1), admits the form

$$u(t,x,y,m) = \lambda(t) \log(x) + \phi(t,y,m),$$

(A.2)

with \(\lambda(t) \equiv 1\) and the function \(\phi\) satisfies the following linear partial differential equation

$$\phi_t + r + [\alpha(y) + \rho \beta(y)m] \phi_y + \frac{1}{2} \left[ \frac{\mu(y)}{\sigma(y)} + m \right]^2 \phi_{yy} + \frac{1}{2} \beta^2(y) \phi_{yy} + \frac{1}{2} f^2(t) \phi_{mm} + \rho \beta(y) f(t) \phi_{ym} = 0,$$

(A.3)
Substituting the above expressions in equation (3.3) leads to the partial differential equation

$$u_t(t,x,y,m) = \lambda(t) \log(x) + \varphi(t,y,m),$$  \hspace{1cm} (A.5)

with boundary condition \( \varphi(T,y,m) = 0 \). In this case, the optimal investment strategy is given in feedback form by

$$\pi^*(t,x,y,m) = \left( \frac{\mu(y)}{\sigma(y)} + m \right) \frac{1}{\sigma(y)},$$  \hspace{1cm} (A.4)

**Proof.** Suppose the partial differential equation (3.3) admits a classical solution \( u \in C^{1,2,2}(\mathbb{S}) \) for every quadruple \((t,x,y,m) \in \mathbb{S}\). We look for solutions using the following ansatz:

$$u(t,x,y,m) = \lambda(t) \log(x) + \varphi(t,y,m),$$  \hspace{1cm} (A.5)

where \( \varphi(t,y,m) \) is a suitable function with boundary conditions \( \varphi(T,y,m) = 0 \) (this follows from the boundary condition \( u(T,x,y,m) = U(x) \)), which will be determined later. Differentiating equation (A.5) with respect to \((t,x,y,m)\), yields

\[
\begin{align*}
    u_t(t,x,y,m) &= \lambda'(t) \log(x) + \varphi_t, \\
    u_x(t,x,y,m) &= \lambda(t) / x, \\
    u_m(t,x,y,m) &= \varphi_m, \\
    u_{xx}(t,x,y,m) &= -\lambda(t) / x^2, \\
    u_{yy}(t,x,y,m) &= \varphi_{yy}, \\
    u_{xm}(t,x,y,m) &= 0.
\end{align*}
\]

Substituting the above expressions in equation (3.3) leads to the partial differential equation

$$\lambda'(t) \log(x) + \varphi_t + r\lambda(t) + [\alpha(y) + \rho \beta(y) m] \varphi_y + \frac{1}{2} \left[ \frac{\mu(y)}{\sigma(y)} + m \right]^2 \lambda(t) + \frac{1}{2} \beta^2(y) \varphi_{yy} + \frac{1}{2} f^2(t) \varphi_{mm} + \rho \beta(y) f(t) \varphi_{ym} = 0,$$

which can be split into two equations, namely

\[
\begin{align*}
    \lambda'(t) &= 0, \\
    \lambda(T) &= 1, \\
\end{align*}
\]

(A.7)

and

\[
\begin{align*}
    \varphi_t + r\lambda(t) + [\alpha(y) + \rho \beta(y) m] \varphi_y + \frac{1}{2} \left[ \frac{\mu(y)}{\sigma(y)} + m \right]^2 \lambda(t) + \frac{1}{2} \beta^2(y) \varphi_{yy} + \frac{1}{2} f^2(t) \varphi_{mm} + \rho \beta(y) f(t) \varphi_{ym} &= 0, \\
\end{align*}
\]

(A.8)

with boundary condition \( \varphi(T,y,m) = 0 \). At first, we notice that the solution of the ordinary differential equation (A.7) is \( \lambda(t) \equiv 1 \) and then, by substituting this solution back in the partial differential equation (A.8), we arrive at the linear partial differential equation (A.3). Finally, by substituting the trial solution (A.5) in equation (3.4) gives the optimal control law (A.4). This concludes the proof.

In view of the specific example of Section 4.1, we have the next result.

**Theorem A.2** Assume logarithmic preferences (equation (A.1)) and moreover that the risky asset evolves according to the stochastic differential equations (4.7) and (4.8) and that the information signal
is given by equation (4.9). The value function for the informed economic agent which is also the solution of the stochastic optimal control problem (3.1), admits the form

\[ u(t, x, y, m) = \lambda(t) \log(x) + \varphi(t, y, m), \]  

(A.9)

where \( \lambda(t) \equiv 1 \), and

\[ \varphi(t, y, m) = B_1(t)m^2 + B_2(t)m + B_3(t)y^2 + B_4(t)y + B_5(t)my + B_6(t), \]  

(A.10)

with

\[ B_1(t) = \frac{1}{2}(T - t) \]  

(A.11)

\[ B_2(t) = \frac{\mu}{\sigma}(T - t) \]  

(A.12)

\[ B_3(t) = B_4(t) = B_5(t) = 0 \]  

(A.13)

\[ B_6(t) = \frac{1}{2} \log \left[ \frac{\lambda^2(T - t) + (1 - \lambda)^2}{(1 - \lambda)^2} \right] + \left[ r + \frac{1}{2} \left( \frac{\mu}{\sigma} \right)^2 \right](T - t) + \frac{(1 - \lambda)^2}{2 \left( \lambda^2(T - t) + (1 - \lambda)^2 \right)} - \frac{1}{2} \]  

(A.14)

In this case, the optimal investment strategy for the informed economic agent is given by the feedback rule

\[ \pi^*(t, x, y, m) = \frac{\mu + \sigma m}{\sigma \langle y \rangle + \delta} \]  

(A.15)

**Proof.** For the special case under investigation, \( f(t) \) is given by equation (4.11) and \( \langle \rho, \mu, \sigma, \alpha, \beta \rangle \) are given in Section 4.1. As a result, equation (A.3), leads to

\[ \varphi_t + r + [\alpha(\theta - y) + \beta m] \varphi_x + \frac{1}{2} \left( \frac{\mu}{\sigma} + m \right)^2 + \frac{1}{2} \beta^2 \varphi_{yy} + \frac{1}{2} \left[ \frac{\lambda^2}{\lambda^2(T - t) + (1 - \lambda)^2} \right]^2 \varphi_{mm} \]

\[ - \left[ \frac{\beta \lambda^2}{\lambda^2(T - t) + (1 - \lambda)^2} \right] \varphi_{ym} = 0. \]  

(A.16)

We conjecture a solution to the above linear parabolic partial differential equation, with the following form

\[ \varphi(t, y, m) = B_1(t)m^2 + B_2(t)m + B_3(t)y^2 + B_4(t)y + B_5(t)my + B_6(t), \]

where \( B_1(t), B_2(t), B_3(t), B_4(t), B_5(t) \) and \( B_6(t) \) are suitable functions to be determined later with boundary conditions \( B_1(T) = B_2(T) = B_3(T) = B_4(T) = B_5(T) = B_6(T) = 0 \) (this follows from the boundary condition \( \varphi(T, y, m) = 0 \)). Substituting this trial solution in equation (A.16) we get the following ordinary differential equations

\[ B_1'(t) + \beta B_5(t) + \frac{1}{2} = 0 \]  

(A.17a)

\[ B_2'(t) + \beta B_4(t) + \alpha \theta B_5(t) + \frac{\mu}{\sigma} = 0 \]  

(A.17b)
where the function \( G \) every quadruple \((x,y,z,w)\) forms by with boundary condition \((A.17a)\) and then solve it (ii) and finally substitute the solutions for equations \((A.17a-d)\) into equation \((A.17f)\) and then solve it. This eventually yields to equation \((A.9)\) which is the optimal value function associated with the stochastic optimal control problem at hand.

Regarding the optimal investment strategy \((A.15)\), it follows directly from equation \((A.4)\) adapted to our specific example.

\section{A.2 The case of the power utility function}

In this part, we consider a power-type utility function, that is, the utility function of the form

\[
U(x) = \frac{x^\eta}{\eta},
\]

where \( \eta < 1 \) is the risk aversion parameter.

\textbf{Theorem A.3} Assume power-type preferences (equation \((A.18)\)). Then, the value function for the informed economic agent which is also the solution of the stochastic optimal control problem \((3.1)\), admits the form

\[
u(t,x,y,m) = \frac{x^\eta}{\eta} G(t,y,m),\]

where the function \( G \) satisfies the following non-linear partial differential equation

\[
\begin{align*}
G_t &+ \left[ r\eta - \frac{1}{2} \left( \frac{\mu(y)}{\sigma(y)} + m \right)^2 \eta \right] G + \left[ \alpha(y) - \frac{\rho \beta(y)m}{\eta - 1} - \frac{\rho \sigma(y)}{\eta} \right] G_y \\
&+ \frac{1}{2} \beta^2(y) G_{yy} - \rho^2 \eta G^2_y G + \frac{1}{2} f^2(t) \left[ G_{mm} - \frac{\eta}{\eta - 1} G_m^2 \right] Gy \\
+ \rho \beta(y) f(t) \left[ G_{sm} - \frac{\eta}{\eta - 1} G_m G_y \right] - f(t) \left( \frac{\mu(y)}{\sigma(y)} + m \right) \eta \eta - 1 G_m = 0,
\end{align*}
\]

with boundary condition \( G(T,y,m) = 1 \). In this case, the optimal investment strategy is given in feedback form by

\[
\pi^*(t,x,y,m) = \left[ - \left( \frac{\mu(y)}{\sigma(y)} + m \right) \frac{1}{\eta - 1} - \rho \beta(y) \frac{G_y}{(\eta - 1)G} - f(t) \frac{G_m}{(\eta - 1)G} \right] \frac{1}{\sigma(y)}.
\]

\textbf{Proof.} Suppose the partial differential equation \((3.3)\) admits a classical solution \( u \in C^{1,2,2,2}(\mathbb{S}) \) for every quadruple \((t,x,y,m)\) \( \in \mathbb{S} \). We look for solutions using the following ansatz:

\[
u(t,x,y,m) = \frac{x^\eta}{\eta} G(t,y,m),
\]
where $G(t,y,m)$ is a suitable function with boundary conditions $G(T,y,m) = 1$ (this follows from the boundary condition $u(T,x,y,m) = U(x)$), which will be determined later. Differentiating the above trial solution with respect to $(t,x,y,m)$, yields

$$
\begin{align*}
&u_t(t,x,y,m) = (x^\eta/y)G_t \
&u_y(t,x,y,m) = (x^\eta/y)G_y \
&u_{y}y(t,x,y,m) = (\eta-1)x^{\eta-2}G_y \
&u_{sy}(t,x,y,m) = x^{\eta-1}G_{yy} \
&u_{ym}(t,x,y,m) = x^\eta G_{ym} \
&u_{mm}(t,x,y,m) = (x^\eta/y)G_{mm}.
\end{align*}
$$

Substituting the above expressions in equation (3.3) leads to the partial differential equation (A.20) and in equation (3.4) gives the optimal control law (A.21). This concludes the proof.

In general, it is very difficult to explicitly solve the non-linear partial differential equation (A.20) for any value of $\rho \in [-1,1]$, and, to the best of our knowledge in this case one has to resort to the theory of viscosity solutions (see e.g. Crandall et al. (1992) or Fleming & Soner (2006)) or follow a numerical approximation scheme, something that is outside of the scope of the present paper. However, in view of the specific example of Section 4.1, we have the next result, in the special case $\rho = 0$.

**THEOREM A.4** Assume power-type preferences (equation (A.18)) and moreover that the risky asset evolves according to the stochastic differential equations (4.7) and (4.8) and that the information signal is given by equation (4.9). The value function for the informed economic agent which is also the solution of the stochastic optimal control problem (3.1), admits the form

$$
u(t,x,y,m) = \frac{x^\eta}{\eta} \exp \{-\eta \Phi(t,y,m)\},$$

where

$$
\Phi(t,y,m) = \Gamma_1(t)m^2 + \Gamma_2(t)m + \Gamma_3(t)y^2 + \Gamma_4(t)y + \Gamma_5(t)my + \Gamma_6(t),
$$

with

$$
\begin{align*}
\Gamma_1(t) &= -\frac{T-t}{2} \frac{\lambda^2(T-t) + (1-\lambda)^2}{\lambda^2(T-t) + (1-\eta)(1-\lambda)^2} \\
\Gamma_2(t) &= -\frac{\mu}{\sigma} (T-t) \frac{\lambda^2(T-t) + (1-\lambda)^2}{\lambda^2(T-t) + (1-\eta)(1-\lambda)^2} \\
\Gamma_3(t) &= \Gamma_4(t) = \Gamma_5(t) = 0 \\
\Gamma_6(t) &= -\frac{1-\eta}{2\eta} \log \left[ \frac{(1-\eta)(1-\lambda)^2}{\lambda^2(T-t) + (1-\eta)(1-\lambda)^2} \right] - \frac{1}{2\eta} \log \left[ \frac{\lambda^2(T-t) + (1-\lambda)^2}{(1-\lambda)^2} \right] \\
&- \frac{\eta}{2} \left[ \frac{\mu^2}{\sigma^2} \frac{(T-t)(1-\lambda)^2}{\lambda^2(T-t) + (1-\eta)(1-\lambda)^2} - \left[ 1 - \frac{\eta - 1}{2(1-\eta)} \left( \frac{\mu}{\sigma} \right)^2 \right] (T-t) \right].
\end{align*}
$$

In this case, The optimal investment strategy for the informed economic agent is given by the feedback rule

$$
\pi^*(t,x,y,m) = \frac{\mu + \sigma m}{\sigma} \frac{\lambda^2(T-t) + (1-\lambda)^2}{\lambda^2(T-t) + (1-\eta)(1-\lambda)^2} \frac{1}{\sigma(\sigma + \delta)}.
$$
Proof: In order to solve equation (A.20) for the special case at hand, we try to fit a solution of the form

\[ G(t, y, m) = \exp \{-\eta \Phi(t, y, m)\}, \]  

(A.29)

where \( \Phi \) is a suitable function with boundary condition \( \Phi(T, y, m) = 0 \) (this follows from the boundary condition \( G(T, y, m) = 1 \)). Differentiating this trial solution with respect to \((t, y, m)\), leads to

\[
\begin{align*}
G_t(t, y, m) &= -\eta \Phi_t \exp \{-\eta \Phi(t, y, m)\} \\
G_y(t, y, m) &= -\eta \Phi_y \exp \{-\eta \Phi(t, y, m)\} \\
G_m(t, y, m) &= -\eta \Phi_m \exp \{-\eta \Phi(t, y, m)\} \\
G_{yy}(t, y, m) &= -\eta \Phi_{yy} \exp \{-\eta \Phi(t, y, m)\} (\Phi_{yy} - \eta \Phi^2) \\
G_{mm}(t, y, m) &= -\eta \exp \{-\eta \Phi(t, y, m)\} (\Phi_{mm} - \eta \Phi^2) \\
G_{ym}(t, y, m) &= -\eta \exp \{-\eta \Phi(t, y, m)\} (\Phi_{ym} - \eta \Phi m).
\end{align*}
\]

Substituting the above expressions in the partial differential equation (A.20), leads to the partial differential equation

\[
\begin{align*}
\Phi_t - r + \frac{1}{2} \left( \frac{\mu(y)}{\sigma(y)} + m \right)^2 - \frac{1}{\eta - 1} + \frac{1}{2} \beta^2(y) [\Phi_{yy} - \eta \Phi_y^2] + \frac{1}{2} f^2(t) \left[ \Phi_{mm} + \frac{\eta - \Phi_m^2}{\eta - 1} \right] \\
- f(t) \left[ \frac{\mu(y)}{\sigma(y)} + m \right] \frac{\eta}{\eta - 1} \Phi_m + \alpha(y)\Phi_y = 0.
\end{align*}
\]

For the special case of the market parameters introduced in Section 4.1, the above partial differential equation, reduces to

\[
\begin{align*}
\Phi_t - r + \frac{1}{2} \left( \frac{\mu(y)}{\sigma(y)} + m \right)^2 - \frac{1}{\eta - 1} + \frac{1}{2} \beta^2 \left[ \Phi_{yy} - \eta \Phi_y^2 \right] + \frac{1}{2} \left[ \frac{\lambda^2}{\lambda^2(T - t) + (1 - \lambda)^2} \right]^2 \left[ \Phi_{mm} + \frac{\eta - \Phi_m^2}{\eta - 1} \right] \\
+ \left[ \frac{\lambda^2}{\lambda^2(T - t) + (1 - \lambda)^2} \right] \left( \frac{\mu(y)}{\sigma(y)} + m \right) \frac{\eta}{\eta - 1} \Phi_m + \alpha(\theta - y)\Phi_y = 0.
\end{align*}
\]

We conjecture a solution to the above partial differential equation, of the form

\[
\Phi(t, y, m) = \Gamma_1(t)m^2 + \Gamma_2(t)m + \Gamma_3(t)y^2 + \Gamma_4(t)y + \Gamma_5(t)my + \Gamma_6(t),
\]

where \( \Gamma_1(t), \Gamma_2(t), \Gamma_3(t), \Gamma_4(t), \Gamma_5(t) \) and \( \Gamma_6(t) \) are suitable functions to be determined with boundary conditions \( \Gamma_1(T) = \Gamma_2(T) = \Gamma_3(T) = \Gamma_4(T) = \Gamma_5(T) = \Gamma_6(T) = 0 \) (this follows from the boundary condition \( \Phi(T, y, m) = 0 \)). By substituting this trial solution, we get the following ordinary differential equations

\[
\begin{align*}
\Gamma_1'(t) + \frac{2\eta}{\eta - 1} f^2(t)\Gamma_1^2(t) - \frac{2\eta}{\eta - 1} f(t)\Gamma_1(t) - \frac{1}{2} \beta^2 \eta^2 \Gamma_2^2(t) + \frac{1}{2(\eta - 1)} &= 0 \quad (A.30a) \\
\Gamma_2'(t) - \frac{2\eta}{\eta - 1} \frac{\mu}{\sigma} f(t)\Gamma_1(t) - \frac{\eta}{\eta - 1} f(t)\Gamma_2(t) + \frac{2\eta}{\eta - 1} f^2(t)\Gamma_1(t)\Gamma_2(t) - \beta^2 \eta^2 \Gamma_4(t)\Gamma_5(t)
\end{align*}
\]
+ \alpha \theta \Gamma_3(t) + \frac{1}{\eta - 1} \frac{\mu}{\sigma} = 0 \quad (A.30b)

\Gamma'_3(t) - 2\beta^2 \eta^2 \Gamma_3(t) + \frac{\eta}{2(\eta - 1)} f^2(t) \Gamma_3(t) = 0 \quad (A.30c)

\Gamma'_4(t) - 2\beta^2 \eta^2 \Gamma_3(t) \Gamma_4(t) + \frac{\eta}{\eta - 1} f^2(t) \Gamma_2(t) \Gamma_5(t) - \frac{\eta}{\eta - 1} \frac{\mu}{\sigma} f(t) \Gamma_5(t) + 2\alpha \theta \Gamma_3(t) - \alpha \Gamma_4(t) = 0 \quad (A.30d)

\Gamma'_5(t) - 2\beta^2 \eta^2 \Gamma_3(t) \Gamma_5(t) + \frac{2\eta}{\eta - 1} f^2(t) \Gamma_1(t) \Gamma_5(t) - \frac{\eta}{\eta - 1} f(t) \Gamma_5(t) - \alpha \Gamma_5(t) = 0 \quad (A.30e)

\Gamma'_6(t) + f^2(t) \Gamma_4(t) - \frac{\eta}{\eta - 1} \frac{\mu}{\sigma} f(t) \Gamma_2(t) + \frac{\eta}{2(\eta - 1)} f^2(t) \Gamma_2(t) + \beta^2 \Gamma_3(t) + \alpha \theta \Gamma_4(t) - r + \frac{1}{2(\eta - 1)} \left( \frac{\mu}{\sigma} \right)^2 - \frac{1}{2} \beta^2 \eta^2 \Gamma_2^2(t) = 0. \quad (A.30f)

In order to solve the above system of ordinary differential equations, we proceed as follows: (i) we solve the ordinary differential equation (A.30e) (ii) we substitute this solution in equations (A.30a) and (A.30c) and solve them (iii) we substitute the solutions of (A.30c) and (A.30e) in equation (A.30d) and solve it (iv) we substitute the solutions of (A.30d) and (A.30e) in equation (A.30b) and finally, the solution of (A.30f) follows.

Regarding the optimal investment strategy, from equation (A.21) adapted to the specific example at hand, we observe that

$$
\pi^*(t,x,y,m) = \left[ - \left( \frac{\mu(y)}{\sigma^2(y)} + m \right) \frac{1}{\eta - 1} - \rho \beta(y) \frac{G_y}{(\eta - 1)G} - f(t) \frac{G_m}{(\eta - 1)G} \right] \frac{1}{\sigma(y)}
$$

$$
= \left[ - \left( \frac{\mu}{\sigma} + m \right) \frac{1}{\eta - 1} + \rho \frac{\eta}{\eta - 1} \frac{\lambda}{\eta - 1} \frac{\lambda^2}{\lambda^2(T - t) + (1 - \lambda)^2} \frac{\eta}{\eta - 1} \frac{\Phi_m}{\Phi} \right] \frac{1}{\sigma(|y| + \delta)}
$$

$$
= \left[ - \left( \frac{\mu}{\sigma} + m \right) \frac{1}{\eta - 1} + \frac{\lambda^2}{\lambda^2(T - t) + (1 - \lambda)^2} \frac{\eta}{\eta - 1} \frac{\Phi_m}{\Phi} \right] \frac{1}{\sigma(|y| + \delta)}
$$

$$
= \frac{\mu}{\sigma} \frac{\lambda^2}{\lambda^2(T - t) + (1 - \lambda)^2} \frac{1}{\sigma(|y| + \delta)},
$$

which follows directly from (A.24), (A.25). This completes the proof.

B. The case of multiple assets and factors

Let us consider the filtered probability space \((\Omega, \mathcal{F}, \mathbb{P})\) that satisfies the usual hypotheses of right continuity and completeness, where \(\mathcal{F}_t = \sigma(W(s), s \leq t)\) is the natural filtration induced by the \(d\)-dimensional Brownian motion \(W(t)\). Here, the superscript \(\top\) denotes transpose of a matrix or a vector.
On the finite time horizon $[0, T]$, with $T \in (0, +\infty)$, we consider a financial market consisting of $d + 1$ investment opportunities, namely

- a risk free asset (e.g. bank account), with dynamics described by

$$\frac{dS_0(t)}{S_0(t)} = rd_t,$$

where $r > 0$ denotes the interest rate.

- $d$ risky assets, whose price process at time $t \in [0, T]$ is denoted by $S(t) = (S_1(t), \ldots, S_d(t))^\top$ and evolves according to the stochastic differential equation

$$dS(t) = \text{diag}(S(t)) \left( [r1 + \mu(Y(t))]dt + \sigma(Y(t))dW(t) \right),$$

where $\mu(y) \in \mathbb{R}^d$ is the vector of instantaneous mean rates of return, $\sigma(y) \in \mathbb{R}^{d \times d}$ is the instantaneous volatility matrix, $1 = (1, \ldots, 1)^\top \in \mathbb{R}^d$ and $\text{diag}(S(t)) \in \mathbb{R}^{d \times d}$ is a diagonal matrix with the vector $S(t)$ in the diagonal and zeros off the diagonal.

Here, $Y$ is an $\mathbb{R}^n$-valued external stochastic factor process that effects the price of the risky assets and it evolves dynamically according to the stochastic differential equation

$$dY(t) = \alpha(Y(t))dt + \beta(Y(t))dW(t),$$

where $\alpha(y) \in \mathbb{R}^n$ and $\beta(y) \in \mathbb{R}^{n \times d}$ denotes the volatility matrix for the factor process.

**Remark B.1** A possible extension would be to include jumps in the stochastic differential equation (B.2) that describes the evolution of the risky asset. This case, albeit mathematically interesting, requires a completely different approach than the one undertaken in the present paper. To the best of our knowledge, the greatest challenge would be to deal with a resulting integrodifferential fully nonlinear equation, associated with the control problem at hand, which, to the best of our knowledge, can only be treated numerically or in terms of viscosity solutions (for more information on viscosity solutions, see, e.g. Fleming & Soner (2006)). Concerning the inclusion of jumps within an inside information framework, the interested reader is referred to the paper of David & Okur (2009) for the characterization of the optimal investment choice.

**Assumption B.1** We assume that

(i) The function $\ell = (\mu, \sigma, \alpha, \beta)$ is sublinear and globally Lipschitz, that is, for every $y, \bar{y} \in \mathbb{R}^n$ there exists some constant $C > 0$, such that

$$\|\ell(y) - \ell(\bar{y})\| \leq C\|y - \bar{y}\|,$$

and

$$\|\ell(y)\| \leq C(1 + \|y\|^2).$$

(ii) The matrix $\sigma(\cdot)$ is invertible for every $y \in \mathbb{R}^n$.

(iii) $\|\sigma(y)\sigma^\top(y)\|, \|\beta(y)\beta^\top(y)\| \in L^1$. 

(iv) For each $R > 0$, there exist constants $\mu_1(R), \mu_2(R) > 0$ depending on $R$, such that for $x, \eta \in \mathbb{R}^n$, $||x|| \leq R, \xi \in \mathbb{R}^d$,

$$
\mu_1(R)||\xi||^2 \leq \xi^T \sigma \sigma^T (x) \xi \leq \mu_2(R)||\xi||^2
$$

$$
\mu_1(R)||\eta||^2 \leq \eta^T \beta \beta^T (x) \eta \leq \mu_2(R)||\eta||^2.
$$

**Remark B.2** Assumption B.1(i) is needed in order to guarantee the existence of a unique strong solution for the system of stochastic differential equations (B.2) and (B.3). This is somewhat standard in the relative literature and can be relaxed in numerous ways. The rest of Assumption B.1 gives the non-degeneracy of the volatility covariance matrix $\sigma \sigma^T$ and the factor covariance matrix $\beta \beta^T$ and is needed for the well-posedness of the problem and the associated HJB equation we will derive later, see for example Hata & Sheu (2012).

We envision a portfolio manager, who, at time $t \in [0, T]$ invests proportion of her wealth $\pi(t) = (\pi_1(t), \ldots, \pi_d(t))$ in the risky market. The remaining proportion, $1 - \pi^T(t)1$ is invested in the bank account. Hence, by taking into account equations (B.1)-(B.3), the wealth of the manager is governed by the dynamics

$$
\frac{dX(t)}{X(t)} = \pi^T(t) \text{diag}(S(t))^{-1} dS(t) + (1 - \pi^T(t)1) \frac{dS_0(t)}{S_0(t)}
$$

$$
= \left[ r + \pi^T(t) \mu(Y(t)) \right] dt + \pi^T(t) \sigma(Y(t)) dW(t).
$$

**Definition B.1** Let $\mathcal{F}$ be a general filtration. We denote by $\mathcal{A}(\mathcal{F}; T)$ the class of admissible strategies $\pi(t)$ that satisfy the following conditions:

(i) $\pi(t) : [0, T] \times \Omega \to \Theta \subset \mathbb{R}^d$ is a progressively measurable mapping with respect to the filtration $\mathcal{F}$, where $\Theta$ is a compact subset of $\mathbb{R}^d$.

(ii) $\mathbb{E} \left[ \int_0^T ||\sigma(Y(t))\pi(t)||^2 dt \right] < \infty$, $\mathbb{P}$-a.s.;

(iii) The SDE (B.4) admits a unique strong solution.

Moreover, we assume that the manager, from the beginning of the trading interval $[0, T]$, observes a noisy signal $\Psi$ (d-dimensional vector) concerning the terminal values of the components of the Brownian motion $W$. This signal is not precise, in the sense it is subject to some observation noise. As a result, the manager makes her decisions based on the information contained in the filtration

$$
\mathcal{H}_t = \mathcal{F}_t \vee \sigma(\Psi)
$$

$$
= \sigma(W(s), s \leq t) \vee \sigma(\Psi), t \in [0, T],
$$

which is defined so as to satisfy the usual hypotheses of right continuity and completeness. Furthermore, by following a similar argument as in Section 2.3, we assume that the nature of the signal $\Psi$, allows, under the enlarged filtration, the semimartingale decomposition

$$
W(t) = \tilde{W}(t) + \int_0^t k(s),
$$

(B.5)

where $\tilde{W}$ is a Brownian motion under $\mathcal{H} = (\mathcal{H}_t)_{t \in [0, T]}$ and $k(t) := k(t, W(t); \lambda, T, \Psi)$ is a $\mathcal{H}_t$-measurable stochastic process, known as the information drift. Thus, we have placed ourselves within the classical context of the theory of initial enlargement of filtrations described in Section 2.3.
ASSUMPTION B.2 The information drift satisfies the integrability condition

$$\mathbb{E} \left[ \int_0^T \|k(s)\|^2 ds \right] < \infty.$$  

The information drift, defined in equation (B.5), is a stochastic process. Hence, in order to be able to include dynamic programming arguments in our framework, it is important to know its dynamics. However, since the exact form of the information signal $\Psi$ is not available yet, we have to make an appropriate guess about its dynamics. In this vein, a plausible assumption for the dynamics of $k(t)$ is the following.

ASSUMPTION B.3 The information drift evolves according to the following stochastic differential equation

$$dk(t) = \Lambda(t)d\tilde{W}(t)$$

$$k(0) = k_0(T, \Psi) \in \mathbb{R}^d.$$  

(B.6)

where $\Lambda(t) : \mathbb{R}_+ \to \mathbb{R}^{d \times d}$ is an appropriate matrix function.

PROPOSITION B.1 The wealth of the informed manager, under the observation of the information signal $\Psi$, evolves according to the dynamics

$$\frac{dX(t)}{X(t)} = \left[ r + \pi^\top(Y(t)) + \pi^\top(Y(t))\sigma(Y(t))k(t) \right] dt + \pi^\top(Y(t))\sigma(Y(t))d\tilde{W}(t)$$

$$dY(t) = [\alpha(Y(t)) + \beta(Y(t))k(t)] dt + \beta(Y(t))d\tilde{W}(t)$$

$$dk(t) = \Lambda(t)d\tilde{W}(t).$$  

(B.7)

Proof. The proof follows by substituting the semimartingale decomposition (B.5) in the stochastic differential equations (B.4) and (B.3) and by taking into consideration equation (B.6) \qed

The aim of the informed manager, is to maximize the expected utility of her terminal wealth by taking explicitly into account the information signal she possesses, that is, she faces the problem

$$J(t,x,y,k) = \sup_{\pi \in \mathcal{A}[\mathbb{H};T]} \mathbb{E} \left[ U(X^\pi(T)) \mid \Psi \right],$$  

subject to the state process (B.7).

REMARK B.3 At this point and before proceeding any further in our analysis, we should emphasize that (i) the information drift process $k(t)$ is adapted to the observed (enlarged) filtration $\mathbb{H}$. As a result, once the information signal $\Psi$ is available, $k(t)$ is known to the agent (ii) even though the stochastic factor process cannot be traded directly, it is assumed to be observed at any time time $t \in [0,T]$. As a result, we have placed ourselves under the necessary markovian umbrella that enables the application of dynamic programming techniques.

If the value function $J$ is smooth enough, then one can show that is satisfies the Hamilton-Jacobi-Bellman equation:
J_t + \sup_{\pi_t} \left( \left[ \pi^T \mu(y) + \pi^T \sigma(y) k \right] x J_x + \frac{1}{2} \pi^T \sigma(y) \sigma^T(y) \pi x^2 J_{xx} + x \pi^T \sigma(y) \beta^T(y) J_{xy} \\
+ x \pi^T \sigma(y) \Lambda(t)^T J_{sk} \right) + r x J_x + \left[ a^T(y) + \beta^T(y) k^T \right] J_y + \frac{1}{2} tr \left( \beta(y) \beta^T(y) J_{yy} \right) \\
+ \frac{1}{2} tr \left( \Lambda(t) \Lambda(t)^T J_{kk} \right) + \beta(y) \Lambda(t)^T J_{sk} = 0,

where

\begin{align*}
J_y &= (J_{y1}, \ldots, J_{yn})^T \\
J_k &= (J_{k1}, \ldots, J_{kn})^T \\
J_{xy} &= (J_{xy1}, \ldots, J_{xyn})^T \\
J_{yk} &= (J_{yk1}, \ldots, J_{ykn})^T.
\end{align*}

**Theorem B.1** Consider the fully nonlinear partial differential equation

\begin{align*}
J_t + r x J_x + \left[ a^T(y) + \beta^T(y) k^T \right] J_y + \frac{1}{2} tr \left( \beta(y) \beta^T(y) J_{yy} \right) + \frac{1}{2} tr \left( \Lambda(t) \Lambda(t)^T J_{kk} \right) + \beta(y) \Lambda(t)^T J_{sk} \\
- \frac{1}{2} \left[ \mu^T(y) (\sigma(y) \sigma^T(y))^{-1} \mu(y) + k^T k + k^T \mu(y) \sigma(y)^{-1} + \mu^T(y) k \left( \sigma^T(y) \right)^{-1} \right] \frac{J_{xx}}{J_{xx}} \\
- \frac{1}{2} \left[ k^T \beta^T(y) + \beta(y) \mu(y) \sigma(y)^{-1} + \mu^T(y) \beta^T(y) \left( \sigma^T(y) \right)^{-1} + \beta(y) k \right] J_{yy} \frac{1}{J_{yy}} \\
- \frac{1}{2} \left[ \mu^T(y) \Lambda(t)^T + k^T \Lambda(t)^T + \Lambda(t) k + \Lambda(t) \mu(y) \sigma(y)^{-1} \right] J_{sk} \frac{1}{J_{xx}} \\
- \frac{1}{2} J_{yy} \beta(y) \beta^T(y) J_{xy} \frac{1}{J_{xx}} - \frac{1}{2} J_{yy} \Lambda(t) \Lambda(t)^T J_{sk} \frac{1}{J_{xx}} - \frac{1}{2} \left[ \beta(y) \Lambda(t)^T + \Lambda(t) \beta^T(y) \right] J_{ym} J_{yy} \frac{1}{J_{xx}} = 0
\end{align*}

(B.10)

If (B.10) admits a smooth solution $J$ and furthermore the function

$$
\pi^*(t,x,y,k) = \frac{J_{k}}{x J_{xx}} \mu(y) \left[ \sigma(y) \sigma^T(y) \right]^{-1} - \frac{J_{k}}{x J_{xx}} k \left[ \sigma^T(y) \right]^{-1} - \beta^T(y) \left[ \sigma^T(y) \right]^{-1} \frac{J_{yy}}{x J_{xx}}
$$

(B.11)

provides an admissible control law, then $J$ is the value function for the control problem and $\pi^*$ is the optimal feedback control law.

**Proof.** For each quadruple $(t,x,y,k) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^d := S_1$, we must solve the equation

\begin{align*}
J_t + \max_{\pi} \left( \left[ \pi^T \mu(y) + \pi^T \sigma(y) k \right] x J_x + \frac{1}{2} \pi^T \sigma(y) \sigma^T(y) \pi x^2 J_{xx} + x \pi^T \sigma(y) \beta(y)^T J_{xy} \\
+ x \pi^T \sigma(y) \Lambda(t)^T J_{sk} \right) + r x J_x + \left[ a^T(y) + \beta(y) k^T \right] J_y + \frac{1}{2} tr \left( \beta(y) \beta^T(y) J_{yy} \right) \\
+ \frac{1}{2} tr \left( \Lambda(t) \Lambda(t)^T J_{kk} \right) + \beta(y) \Lambda(t)^T J_{sk} = 0,
\end{align*}

(B.12)

with boundary condition $J(T,x,y,m) = U(x)$. Assume that the maximum in equation (B.12) is attained in the interior of the control region (assuming of course that the control set $\Theta$ has a non-empty interior...
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\[ \pi^*(t,x,y,m) \in \Theta^0 \text{ for all } (t,x,y,m) \in \mathbb{S}. \]

Differentiating the above expression with respect to \( \pi \) and setting the derivative equal to zero, gives the candidate optimal control

\[
\hat{\pi}(t,x,y,k) = -\left( \frac{J_x}{xJ_{xx}} \right) \mu(y) \left[ \sigma(y) \sigma(y)^\top \right]^{-1} - \left( \frac{J_y}{xJ_{xx}} \right) \left[ \sigma(y) \sigma(y)^\top \right]^{-1} \beta(y)^\top \left[ \sigma(y) \sigma(y)^\top \right]^{-1} \frac{J_y}{xJ_{xx}} - \Lambda(t)^\top \left[ \sigma(y) \sigma(y)^\top \right]^{-1} \frac{J_{sk}}{xJ_{xx}}.
\]

(B.13)

In this case, if we place this expression back in equation (B.12), we arrive to the nonlinear partial differential equation (B.10). Assume that the partial differential equation (B.10) admits a classical solution \( \hat{J} \in C^{1,2,2,2}(\mathbb{S}_1) \) that satisfies the differentiability conditions \( \hat{J}_{xx}, \hat{J}_{yy}, \hat{J}_{kk} < 0 \) and \( \hat{J}_x, \hat{J}_y, \hat{J}_k > 0. \) By substituting this solution back in equation (B.13) leads to equation (B.11). If moreover \( \hat{\pi}(t,x,y,k) \in \Theta^0, \) then \( \pi^*(t,x,y,k) \) coincides with \( \hat{\pi}(t,x,y,k). \) The rest is a straightforward application of the verification theorem and heavily relies on Itô’s lemma and standard arguments. For a complete proof, we refer the interested reader to Chapter 3 of Pham (2009)

References


