The Incompetence Trap:
The (Conditional) Irrelevance of Agency Expertise

ONLINE APPENDIX

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Expected Value of ω

As $\omega \sim U[-\Omega, \Omega]$, the marginal distribution $f_1(\omega)$ is as follows:

$$f_1(\omega) = \begin{cases} \frac{1}{2\Omega} & \text{if } -\Omega \leq \omega \leq \Omega \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, as $\hat{\omega} \sim U[c\omega - \Omega(1-c), c\omega + \Omega(1-c)]$, the conditional distribution of $\hat{\omega}$ given ω is as follows:

$$g_2(\hat{\omega}\,|\omega) = \begin{cases} \frac{1}{2\Omega - 2c\Omega} & \text{if } (1-c)(-\Omega) + c\omega \leq \hat{\omega} \leq c\omega + (1-c)(\Omega) \\ 0 & \text{otherwise.} \end{cases}$$

The joint distribution of the two is therefore:

$$f(\omega, \hat{\omega}) = \begin{cases} \frac{1}{4\Omega^2 - 4c\Omega^2} & \text{if } (1-c)(-\Omega) + c\omega \leq \hat{\omega} \leq c\omega + (1-c)(\Omega) \\ & \text{and } -\Omega \leq \omega \leq \Omega \\ 0 & \text{otherwise.} \end{cases}$$

The conditional distribution $g_1(\omega|\,\hat{\omega})$ and marginal distribution $g_2(\hat{\omega})$ will depend on the value of $\hat{\omega}$ relative to c and Ω.

Case 1: Suppose $-\Omega < \frac{\Omega c - \Omega + \hat{\omega}}{c} \leq \omega \leq \frac{\Omega - \Omega c + \hat{\omega}}{c} < \Omega$. Then it must hold that $c > \max\left\{\frac{1}{2} + \frac{\hat{\omega}}{2\Omega}, \frac{1}{2} - \frac{\hat{\omega}}{2\Omega}\right\}$. Then the marginal distribution $g_2(\hat{\omega})$ is as follows:

$$g_2(\hat{\omega}) = \begin{cases} \frac{1}{4c\Omega - 2\Omega} & \text{if } \Omega - 2c\Omega \leq \hat{\omega} \leq 2c\Omega - \Omega \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, the conditional distribution $g_1(\omega|\,\hat{\omega})$ is s follows:

$$g_1(\omega|\,\hat{\omega}) = \begin{cases} \frac{c}{2\Omega - 2\Omega c} & \text{if } \frac{\Omega c - \Omega + \hat{\omega}}{c} \leq \omega \leq \frac{\Omega - \Omega c + \hat{\omega}}{c} \\ 0 & \text{otherwise.} \end{cases}$$

The joint distribution $f(\omega, \hat{\omega})$ is thus:

$$\begin{aligned} f(\omega, \hat{\omega}) &= g_1(\omega|\,\hat{\omega}) g_2(\hat{\omega}) \\ &= \begin{cases} \frac{c}{4(3c - 2c^2 - 1)\Omega^2} & \text{if } \frac{\Omega c - \Omega + \hat{\omega}}{c} \leq \omega \leq \frac{\Omega - \Omega c + \hat{\omega}}{c} \\ & \text{and } \Omega - 2c\Omega \leq \hat{\omega} \leq 2c\Omega - \Omega \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

In this case, the conditional expectation of ω given the observed value $\hat{\omega}$ is as follows:

$$E[\omega|\hat{\omega}] = \int_{\frac{\Omega c-\Omega+\hat{\omega}}{c}}^{\frac{\Omega-\Omega c+\hat{\omega}}{c}} \frac{\omega c}{2\Omega - 2\Omega c} d\omega = \frac{\hat{\omega}}{c}.$$

Case 2a: Suppose $\frac{\Omega c-\Omega+\hat{\omega}}{c} < -\Omega \leq \omega \leq \frac{\Omega-\Omega c+\hat{\omega}}{c} < \Omega$ and $c \geq \frac{1}{2}$. Then it must hold that $\frac{1}{2} + \frac{\hat{\omega}}{2\Omega} < c < \frac{1}{2} - \frac{\hat{\omega}}{2\Omega}$. From this, the marginal distribution $g_2(\hat{\omega})$ is as follows:

$$g_2(\hat{\omega}) = \begin{cases} \frac{1}{2\Omega - 2c\Omega} & \text{if } -\Omega \leq \hat{\omega} \leq \Omega - 2c\Omega \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, the conditional distribution $g_1(\omega|\hat{\omega})$ is as follows:

$$g_1(\omega|\hat{\omega}) = \begin{cases} \frac{c}{\Omega + \hat{\omega}} & \text{if } -\Omega \leq \omega \leq \frac{\Omega - \Omega c + \hat{\omega}}{c} \\ 0 & \text{otherwise.} \end{cases}$$

The joint distribution $f(\omega, \hat{\omega})$ is thus:

$$\begin{aligned} f(\omega, \hat{\omega}) &= g_1(\omega|\hat{\omega}) g_2(\hat{\omega}) \\ &= \begin{cases} \frac{c}{(2\Omega - 2c\Omega)(\Omega + \hat{\omega})} & \text{if } -\Omega \leq \omega \leq \frac{\Omega - \Omega c + \hat{\omega}}{c} \\ & \quad \text{and } -\Omega \leq \hat{\omega} \leq \Omega - 2c\Omega \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

In this case, the conditional expectation of ω given the observed value $\hat{\omega}$ is as follows:

$$E[\omega|\hat{\omega}] = \int_{-\Omega}^{\frac{\Omega-\Omega c+\hat{\omega}}{c}} \frac{\omega c}{\Omega + \hat{\omega}} d\omega = \frac{\Omega + \hat{\omega}}{2c} - \Omega.$$

Case 2b: Suppose $\frac{\Omega c-\Omega+\hat{\omega}}{c} < -\Omega \leq \omega \leq \frac{\Omega-\Omega c+\hat{\omega}}{c} < \Omega$ and $c \leq \frac{1}{2}$. Then it must hold that $\frac{1}{2} + \frac{\hat{\omega}}{2\Omega} < c < \frac{1}{2} - \frac{\hat{\omega}}{2\Omega}$. From this, the marginal distribution $g_2(\hat{\omega})$ is as follows:

$$g_2(\hat{\omega}) = \begin{cases} \frac{1}{2c\Omega} & \text{if } -\Omega \leq \hat{\omega} \leq 2c\Omega - \Omega \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, the conditional distribution $g_1(\omega|\hat{\omega})$ is as follows:

$$g_1(\omega|\hat{\omega}) = \begin{cases} \dfrac{c}{\Omega+\hat{\omega}} & \text{if } -\Omega \leq \omega \leq \dfrac{\Omega-\Omega c+\hat{\omega}}{c} \\ 0 & \text{otherwise.} \end{cases}$$

The joint distribution $f(\omega,\hat{\omega})$ is thus:

$$\begin{aligned} f(\omega,\hat{\omega}) &= g_1(\omega|\hat{\omega})g_2(\hat{\omega}) \\ &= \begin{cases} \dfrac{c}{(2c\Omega)(\Omega+\hat{\omega})} & \text{if } -\Omega \leq \omega \leq \dfrac{\Omega-\Omega c+\hat{\omega}}{c} \\ & \text{ and } -\Omega \leq \hat{\omega} \leq 2c\Omega-\Omega \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

In this case, the conditional expectation of ω given the observed value $\hat{\omega}$ is as follows:

$$\begin{aligned} E[\omega|\hat{\omega}] &= \int_{-\Omega}^{\frac{\Omega-\Omega c+\hat{\omega}}{c}} \frac{\omega c}{\Omega+\hat{\omega}} d\omega \\ &= \frac{\Omega+\hat{\omega}}{2c} - \Omega. \end{aligned}$$

Case 3a: Suppose $-\Omega < \frac{\Omega c-\Omega+\hat{\omega}}{c} \leq \omega \leq \Omega < \frac{\Omega-\Omega c+\hat{\omega}}{c}$ and $c \geq \frac{1}{2}$. Then it must hold that $\frac{1}{2} - \frac{\hat{\omega}}{2\Omega} < c < \frac{1}{2} + \frac{\hat{\omega}}{2\Omega}$. From this, the marginal distribution $g_2(\hat{\omega})$ is as follows:

$$g_2(\hat{\omega}) = \begin{cases} \dfrac{1}{2\Omega-2c\Omega} & \text{if } 2c\Omega-\Omega \leq \hat{\omega} \leq \Omega \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, the conditional distribution is

$$g_1(\omega|\hat{\omega}) = \begin{cases} \dfrac{c}{\Omega-\hat{\omega}} & \text{if } \dfrac{\Omega c-\Omega+\hat{\omega}}{c} \leq \omega \leq \Omega \\ 0 & \text{otherwise.} \end{cases}$$

The joint distribution $f(\omega,\hat{\omega})$ is thus:

$$\begin{aligned} f(\omega,\hat{\omega}) &= g_1(\omega|\hat{\omega})g_2(\hat{\omega}) \\ &= \begin{cases} \dfrac{c}{(2\Omega-2c\Omega)(\Omega-\hat{\omega})} & \text{if } \dfrac{\Omega c-\Omega+\hat{\omega}}{c} \leq \omega \leq \Omega \\ & \text{ and } 2c\Omega-\Omega \leq \hat{\omega} \leq \Omega \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

In this case, the conditional expectation of ω given the observed value $\hat{\omega}$ is as follows:

$$\begin{aligned} E[\omega|\hat{\omega}] &= \int_{\frac{\Omega c-\Omega+\hat{\omega}}{c}}^{\Omega} \frac{\omega c}{\Omega-\hat{\omega}} d\omega \\ &= \Omega + \frac{\hat{\omega}-\Omega}{2c}. \end{aligned}$$

Case 3b: Suppose $-\Omega < \frac{\Omega c-\Omega+\hat{\omega}}{c} \leq \omega \leq \Omega < \frac{\Omega-\Omega c+\hat{\omega}}{c}$ and $c \leq \frac{1}{2}$. Then it must hold that $\frac{1}{2} - \frac{\hat{\omega}}{2\Omega} < c < \frac{1}{2} + \frac{\hat{\omega}}{2\Omega}$. From this, the marginal distribution $g_2(\hat{\omega})$ is as follows:

$$g_2(\hat{\omega}) = \begin{cases} \frac{1}{2c\Omega} & \text{if } \Omega - 2c\Omega \leq \hat{\omega} \leq \Omega \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, the conditional distribution is

$$g_1(\omega|\hat{\omega}) = \begin{cases} \frac{c}{\Omega-\hat{\omega}} & \text{if } \frac{\Omega c-\Omega+\hat{\omega}}{c} \leq \omega \leq \Omega \\ 0 & \text{otherwise.} \end{cases}$$

The joint distribution $f(\omega, \hat{\omega})$ is thus:

$$\begin{aligned} f(\omega,\hat{\omega}) &= g_1(\omega|\hat{\omega}) g_2(\hat{\omega}) \\ &= \begin{cases} \frac{c}{(2c\Omega)(\Omega-\hat{\omega})} & \text{if } \frac{\Omega c-\Omega+\hat{\omega}}{c} \leq \omega \leq \Omega \\ & \quad \text{and } \Omega - 2c\Omega \leq \hat{\omega} \leq \Omega \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

In this case, the conditional expectation of ω given the observed value $\hat{\omega}$ is as follows:

$$\begin{aligned} E[\omega|\hat{\omega}] &= \int_{\frac{\Omega c-\Omega+\hat{\omega}}{c}}^{\Omega} \frac{\omega c}{\Omega-\hat{\omega}} d\omega \\ &= \Omega + \frac{\hat{\omega}-\Omega}{2c}. \end{aligned}$$

Case 4: Suppose $\frac{\Omega c-\Omega+\hat{\omega}}{c} < -\Omega \leq \omega \leq \Omega < \frac{\Omega-\Omega c+\hat{\omega}}{c}$. Then it must hold that $c < \min\left\{\frac{1}{2} + \frac{\hat{\omega}}{2\Omega}, \frac{1}{2} - \frac{\hat{\omega}}{2\Omega}\right\}$. From this, the marginal distribution $g_2(\hat{\omega})$ is as follows:

$$g_2(\hat{\omega}) = \begin{cases} \frac{1}{2\Omega - 4c\Omega} & \text{if } 2c\Omega - \Omega \leq \hat{\omega} \leq \Omega - 2c\Omega \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, the conditional distribution is

$$g_1(\omega|\hat{\omega}) = \begin{cases} \frac{1}{2\Omega} & \text{if } -\Omega \leq \omega \leq \Omega \\ 0 & \text{otherwise.} \end{cases}$$

The joint distribution $f(\omega, \hat{\omega})$ is thus:

$$\begin{aligned} f(\omega,\hat{\omega}) &= g_1(\omega|\hat{\omega})g_2(\hat{\omega}) \\ &= \begin{cases} \frac{1}{4\Omega^2 - 8c\Omega^2} & \text{if } -\Omega \leq \omega \leq \Omega \\ & \quad \text{and } 2c\Omega - \Omega \leq \hat{\omega} \leq \Omega - 2c\Omega \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

In this case, the conditional expectation of ω given the observed value $\hat{\omega}$ is as follows:

$$\begin{aligned} E[\omega|\hat{\omega}] &= \int_{-\Omega}^{\Omega} \frac{\omega}{2\Omega} d\omega \\ &= 0. \end{aligned}$$

Putting it all together, conditional on the observed value $\hat{\omega}$, the expected value of ω is given by

$$E[\omega|\hat{\omega}] = \begin{cases} \frac{\hat{\omega}}{c} & \text{if } c \geq \max\left\{\frac{1}{2} + \frac{\hat{\omega}}{2\Omega}, \frac{1}{2} - \frac{\hat{\omega}}{2\Omega}\right\} \\ \frac{\Omega + \hat{\omega}}{2c} - \Omega & \text{if } \frac{1}{2} + \frac{\hat{\omega}}{2\Omega} \leq c \leq \frac{1}{2} - \frac{\hat{\omega}}{2\Omega} \\ \Omega + \frac{\hat{\omega} - \Omega}{2c} & \text{if } \frac{1}{2} - \frac{\hat{\omega}}{2\Omega} \leq c \leq \frac{1}{2} + \frac{\hat{\omega}}{2\Omega} \\ 0 & \text{if } c \leq \min\left\{\frac{1}{2} + \frac{\hat{\omega}}{2\Omega}, \frac{1}{2} - \frac{\hat{\omega}}{2\Omega}\right\}. \end{cases}$$

Expected Utilities Given $E[\omega|\hat{\omega}]$

After observing $\hat{\omega}$, the agency sets a policy $p \in \mathbb{R}$, which it chooses in order to maximize $Eu_A(p|\hat{\omega}) = -(p + E[\omega|\hat{\omega}] - x_A)^2$. Clearly, the agency will set $p^*(\hat{\omega}) = x_A - E[\omega|\hat{\omega}]$. Given this strategy, the expected utility of player i takes the following general form:

$$Eu_i(x_A, c) = \int\int -g_1(\omega|\hat{\omega})g_2(\hat{\omega})(x_A - E[\omega|\hat{\omega}] + \omega - x_i)^2 d\omega d\hat{\omega}.$$

When $c \in \left[\frac{1}{2}, 1\right]$, the expected value is

$$Eu_i(x_A, c) = \left[\int_{-\Omega}^{\Omega - 2c\Omega} \int_{-\Omega}^{\frac{\Omega - \Omega c + \hat{\omega}}{c}} - \frac{c\left(x_A + \Omega - \frac{\Omega + \hat{\omega}}{2c} + \omega - x_i\right)^2}{(2\Omega - 2c\Omega)(\Omega + \hat{\omega})} d\omega d\,\hat{\omega} \right.$$

$$+ \int_{\Omega - 2c\Omega}^{2c\Omega - \Omega} \int_{\frac{\Omega c - \Omega + \hat{\omega}}{c}}^{\frac{\Omega - \Omega c + \hat{\omega}}{c}} - \frac{c\left(x_A - \frac{\hat{\omega}}{c} + \omega - x_i\right)^2}{4(3c - 2c^2 - 1)\Omega^2} d\omega d\,\hat{\omega}$$

$$\left. + \int_{2c\Omega - \Omega}^{\Omega} \int_{\frac{\Omega c - \Omega + \hat{\omega}}{c}}^{\Omega} - \frac{c\left(x_A - \Omega + \frac{\Omega - \hat{\omega}}{2c} + \omega - x_i\right)^2}{(2\Omega - 2c\Omega)(\Omega - \hat{\omega})} d\omega d\,\hat{\omega} \right]$$

$$= -3(x_A - x_i)^2 - \frac{15(1-c)^2\Omega^2}{27c^2}.$$

When $c \in \left(0, \frac{1}{2}\right]$, the expected value is

$$Eu_i(x_A, c) = \left[\int_{-\Omega}^{2c\Omega - \Omega} \int_{-\Omega}^{\frac{\Omega - \Omega c + \hat{\omega}}{c}} - \frac{c\left(x_A + \Omega - \frac{\Omega + \hat{\omega}}{2c} + \omega - x_i\right)^2}{(2c\Omega)(\Omega + \hat{\omega})} d\omega d\,\hat{\omega} \right.$$

$$+ \int_{2c\Omega - \Omega}^{\Omega - 2c\Omega} \int_{-\Omega}^{\Omega} - \frac{(x_A + \omega - x_i)^2}{4\Omega^2 - 8c\Omega^2} d\omega d\,\hat{\omega}$$

$$\left. + \int_{\Omega - 2c\Omega}^{\Omega} \int_{\frac{\Omega c - \Omega + \hat{\omega}}{c}}^{\Omega} - \frac{c\left(x_A - \Omega + \frac{\Omega - \hat{\omega}}{2c} + \omega - x_i\right)^2}{(2c\Omega)(\Omega - \hat{\omega})} d\omega d\,\hat{\omega} \right]$$

$$= -3(x_A - x_i)^2 - \frac{15\Omega^2}{27}.$$

Therefore, after normalizing by dividing by three:

$$Eu_i(x_A, c) = \begin{cases} -(x_A - x_i)^2 - \frac{5(1-c)^2\Omega^2}{27c^2} & \text{if } c \in \left[\frac{1}{2}, 1\right] \\ -(x_A - x_i)^2 - \frac{5\Omega^2}{27} & \text{if } c \in \left(0, \frac{1}{2}\right] \end{cases}$$