Drinfel’d Twisted Superconformal Algebra
and the Structure of Unbroken Symmetries

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We investigate deformed superconformal symmetries on non(anti)commutative (super)spaces from the point of view of the Drinfel’d twisted symmetries. We classify all possible twist elements derived from an abelian subsector of the superconformal algebra. The symmetry breaking caused by the non(anti)commutativity of the (super)spaces is naturally interpreted as the modification of their coproduct emerging from the corresponding twist element. The remaining unbroken symmetries are determined by the commutative properties of those symmetry generators possessing the twist element. We also comment on non-canonically deformed non(anti)commutative superspaces, particularly those derived from the superconformal twist element $F_{SS}$.

§1. Introduction

The study of noncommutative spaces has recently attracted considerable interests, because it is thought that they may provide a fundamental basis for a theory of quantum gravity.4) Superstring theory, which is believed to be the most promising possibility as a consistent theory of quantum gravity, provides a realization of noncommutative space.3) The simplest noncommutative space, a noncommutative plane, possesses a so-called canonical structure among its coordinates expressed as

$$[x^m, x^n] = i\theta^{mn} \neq 0,$$

where $\theta^{mn}$ is a constant noncommutativity parameter. Note that this canonical noncommutativity breaks the Lorentz invariance of the theory. Field theories on such a noncommutative plane have been intensively studied. (See for example Ref. 4 and references therein.)

The space-time noncommutative plane has been generalized to superspaces.5), 6) The supersymmetric counterpart of Eq. (1) is given by

$$\{\theta^\alpha, \theta^\beta\} = C^{\alpha\beta} \neq 0,$$

where $C^{\alpha\beta}$ is a constant supercommutativity parameter. Note that this canonical supercommutativity breaks the Lorentz invariance of the theory. Field theories on such a noncommutative plane have been intensively studied. (See for example Ref. 4 and references therein.)
where $\theta^{\alpha}$ is the fermionic coordinate of the superspace $(x^m, \theta^{\alpha}, \bar{\theta}^{\dot{\alpha}})$. In this case, the fermionic coordinate is not a Grassmann variable, but, instead, it satisfies a Clifford-like algebra. In general, the non(anti)commutativity of (super)spaces breaks a part of the symmetry of the theory. It is known that the non(anti)commutativity of the superspace (2) breaks half of the supersymmetry corresponding to the anti-supercharge $\bar{Q}$ and the Lorentz symmetry $M_{\dot{\alpha}\beta}$.

When the theory possesses superconformal symmetry, the deformation (2) also breaks the dilatation symmetry, R-symmetry and half of the superconformal supersymmetry $\mathcal{S}$. However, in Ref. 8), it is pointed out that a linear combination of the dilatation and R-symmetry is preserved.

In Ref. 9), it is shown that the canonical noncommutative plane can be described by a representation space of the Drinfel’d twisted Poincaré algebra. This suggests a deep relation between quantized (Hopf) algebra and noncommutative geometry. The theory defined on the canonical noncommutative space preserves the Drinfel’d twisted Poincaré symmetry, even though the ordinary Lorentz symmetry is broken. This idea has been extended to supersymmetry and/or conformal symmetry, and there are many applications to field theories, especially a noncommutative theory of gravity.

In a related work, the authors of Ref. 14), proposed alternative structures of twist elements satisfying the twist equation and showed that the proper choice of the twist element leads to a non-canonical noncommutative space, e.g. Lie algebra and a quadratic type of noncommutative space.

In this paper, we study various aspects of broken symmetries on non(anti)commutative (super)spaces in the sense of the Drinfel’d twisted Hopf algebra. Because the symmetry breaking is caused by the twist element, we can systematically classify the broken and unbroken symmetries in the non(anti)commutative (super)spaces in the language of the twisted Hopf algebra. As an example, we clarify the broken and unbroken symmetries on the superspace caused by twist elements that are constructed from the generators of the superconformal algebra.

The organization of this paper is as follows. In §2, the concept of the Drinfel’d twist in the context of Hopf algebra is introduced. In §3, we classify broken and unbroken symmetries on the various non(anti)commutative superspaces. In particular, we focus on the non-canonical type of non(anti)commutative superspaces caused by the superconformal twist elements and classify broken and unbroken symmetries. Section 4 is devoted to the calculation of deformed coproducts, and §5 contains a summary of this paper.

§2. The Drinfel’d twisted Hopf algebra and noncommutative spaces

In this section, we review the twisted Hopf algebra and its representation space. A Hopf algebra $(\mathcal{H}, +, \circ, \iota, \Delta, \epsilon, \gamma; \mathbb{K})$ over a field $\mathbb{K}$ is a unital associative algebra $(\mathcal{H}, +, \circ, \iota; \mathbb{K})$ with the following linear maps:

- **addition**: $+ : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H}$
- **product**: $\circ : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H}$
- **unit**: $\iota : \mathbb{K} \rightarrow \mathcal{H}$
- **coproduct**: $\Delta : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$
- **counit**: $\epsilon : \mathcal{H} \rightarrow \mathbb{K}$
- **antipode**: $\gamma : \mathcal{H} \rightarrow \mathcal{H}$
These maps possess some properties of duality, as we now describe. Using the Sweedler notation, $\Delta(a) = \sum_i a_i^1 \otimes a_i^2 = \bar{a}_i \otimes \bar{a}^i$, for the coproduct, the relations between the algebra and coalgebra

$$\text{coassociativity: } \Delta(\bar{a}_i) \otimes \bar{a}^i = \bar{a}_i \otimes \Delta(\bar{a}^i)$$

$$\Longleftrightarrow (a \circ b) \circ c = a \circ (b \circ c) : \text{associativity}$$

$$\text{counit: } \epsilon(\bar{a}_i) \bar{a}^i = \bar{a}_i \epsilon(\bar{a}^i) = a$$

$$\Longleftrightarrow \iota(k) \circ a = a \circ \iota(k) = ka : \text{unit}$$

$$\text{antipode: } \gamma(\bar{a}_i) \circ \bar{a}^i = \bar{a}_i \circ \gamma(\bar{a}^i) = \iota(\epsilon(a))$$

hold for all $a, b, c \in \mathcal{H}$, $k \in \mathbb{K}$. We denote the unit element of the Hopf algebra by $1$. The product $\circ$ is extended to tensor powers, $\circ : \mathcal{H}^n \times \mathcal{H}^n \rightarrow \mathcal{H}^n$, which is written in concrete form as

$$(a \otimes \cdots \otimes b) \circ (c \otimes \cdots \otimes d) := (a \circ c) \otimes \cdots \otimes (b \circ d) \quad \forall a, b, c, d \in \mathcal{H}. \quad (5)$$

(Hereafter, the symbol $\circ$ will be omitted unless otherwise noted.) We require the maps to satisfy the following relations:

$$\Delta(ab) = \Delta(a) \Delta(b), \quad \epsilon(ab) = \epsilon(a) \epsilon(b), \quad \gamma(ab) = \gamma(b) \gamma(a). \quad (6)$$

As can be seen from the last relations in Eqs. (4) and (6), the antipode behaves like an inverse element.

A pair $(\rho, \mathcal{A})$, consisting of a representation $\rho$ and a space $\mathcal{A}$ determine how $\mathcal{H}$ acts on $\mathcal{A}$, which is called a (left) $\mathcal{H}$-module algebra. We define the action $\triangleright$ by

$$\left( a \otimes \cdots \otimes b \right) \triangleright \left( \phi \otimes \cdots \otimes \psi \right) := \rho(a) \phi \otimes \cdots \otimes \rho(b) \psi, \quad (7)$$

$$(ab) \triangleright \phi = \rho(ab) \phi = \rho(a) \rho(b) \phi = a \triangleright (b \triangleright \phi), \quad (8)$$

for all $a, b \in \mathcal{H}$ and $\phi, \psi \in \mathcal{A}$. In addition, we require an algebraic structure on $\mathcal{A}$, i.e. the multiplication denoted by $\mu(\phi \otimes \psi) =: \phi \cdot \psi$, which is compatible with a coproduct on the Hopf algebra:

$$a \triangleright \mu(\phi \otimes \psi) = a \triangleright (\phi \cdot \psi) = (\bar{a}_i \triangleright \phi) \cdot (\bar{a}^i \triangleright \psi) = \mu(\Delta(a) \triangleright (\phi \otimes \psi)). \quad (9)$$

For more details, see Ref. 2).

We can systematically construct a modified Hopf algebra $(\mathcal{H}, +, \circ, \iota, \Delta, \epsilon, \gamma; \mathbb{K})$ from an arbitrary, given Hopf algebra $(\mathcal{H}, +, \circ, \iota, \Delta, \epsilon, \gamma; \mathbb{K})^1$ as we now demonstrate. First, assume that there exists an invertible element $F = \sum_i f_i^1 \otimes f_i^2 \in \mathcal{H} \otimes \mathcal{H}$, called a twist element, which satisfies the following relations:

$$F_{12} \left( \Delta \otimes \text{id} \right) F = F_{23} \left( \text{id} \otimes \Delta \right) F \quad \left( F_{12} := F \otimes 1, \quad F_{23} := 1 \otimes F \right), \quad (10)$$

$$\left( \epsilon \otimes \text{id} \right) F = \left( \text{id} \otimes \epsilon \right) F = 1. \quad (11)$$

Then the new maps of the modified algebra are given by

$$\Delta_t := \text{Ad}_F \Delta, \quad \Delta_t(a) = F \Delta(a) F^{-1}, \quad (12)$$

$$\gamma_t := \text{Ad}_U \gamma, \quad \gamma_t(a) = U \gamma(a) U^{-1} \quad \forall a \in \mathcal{H}. \quad (13)$$
Here, $F^{-1} \in \mathcal{H} \otimes \mathcal{H}$ is the inverse of $F$ satisfying $FFF^{-1} = F^{-1}F = 1 \otimes 1$ and $U := \sum f^i_L \gamma(f^i_R)$. All other maps unchanged. This Hopf algebra is referred to as the Drinfel’d twisted or merely the twisted Hopf algebra. A representation space of the twisted Hopf algebra $\mathcal{A}_t$ is derived from the structure of a given $\mathcal{H}$-module $\mathcal{A}$, along with the manner in which the twisted $\mathcal{H}$ is constructed from the untwisted $\mathcal{H}$. The representation $\rho$ of $\mathcal{H}$ is the same, because the twist element does not affect the algebraic maps $(+, \circ)$. The only difference regards the multiplication on the module algebra,

$$\phi \star \psi = \mu_t(\phi \otimes \psi) := \mu(F^{-1} \triangleright (\phi \otimes \psi)). \quad (14)$$

This twisted multiplication $\mu_t$ is compatible with the twisted coproduct $\Delta_t$,

$$a \triangleright \mu_t(\phi \otimes \psi) = \mu(\Delta(a) \triangleright F^{-1} \triangleright (\phi \otimes \psi)) = \mu(F^{-1} \triangleright \Delta_t(a) \triangleright (\phi \otimes \psi)) = \mu_t(\Delta_t(a) \triangleright (\phi \otimes \psi)). \quad (15)$$

A universal enveloping algebra is an example of a Hopf algebra. The universal enveloping Poincaré algebra is frequently considered in connection with the non-commutative plane.\(^9\),\(^10\),\(^17\) Generally speaking, Lie algebras are neither unital nor associative. The structure of the Hopf algebra requires both. Therefore we should regard a Lie algebra $g$ as a vector space and introduce the formal product on it with the unit element defined as the zeroth power of the product, as follows:

$$u_1 \circ \cdots \circ u_n := u_1 \cdots u_n \in \mathfrak{g}^n \quad (u_i \in \mathfrak{g}, \quad n \geq 2), \quad 1 \in \mathfrak{g}^0 \simeq \mathbb{C}. \quad (16)$$

The direct sum of all ranks of the tensor products, $T(g) = \bigoplus_{n=0}^{\infty} \mathfrak{g}^n$, is called a tensor algebra. The tensor algebra $T(g)$ has no Lie algebraic structure. To define the Lie algebraic property, we impose the condition that the commutator can be replaced by the Lie bracket,

$$u \circ v (w x - x w) y \cdots z \sim u \circ v [w, x] y \cdots z \quad \forall u, v, w, x, y, z \in \mathfrak{g}. \quad (17)$$

Mathematically, the ideal $\mathcal{I}$ generated by $uv - vu - [u, v]$ for all $u, v \in \mathfrak{g}$ gives the universal enveloping algebra as a quotient space, $\mathcal{U}(g) = T(g)/\mathcal{I}$.

Let us consider a concrete example. The Poincaré algebra $\mathcal{P}$ consists of the translation generators $P_m$ and the Lorentz generators $M_{mn}$, with the following commutation relations:

$$[P_m, P_n] = 0,$$

$$[M_{mn}, M_{kl}] = -i\eta_{mk}M_{nl} + i\eta_{nk}M_{ml} + i\eta_{ml}M_{nk} - i\eta_{nl}M_{mk},$$

$$[M_{mn}, P_k] = -i\eta_{mk}P_n + i\eta_{nk}P_m. \quad (18)$$

Here, $\eta_{mn} = \text{diag}(-, +, +, +)$ represents the Minkowski metric. The Poincaré algebra $\mathcal{P}$ becomes a Hopf algebra through the definitions

$$\mathcal{H} = \mathcal{U}(\mathcal{P}), \quad \mathbb{K} = \mathbb{C},$$

$$\Delta(u) := u \otimes 1 + 1 \otimes u, \quad \epsilon(u) := 0, \quad \gamma(u) := -u \quad \forall u \in \mathcal{P},$$

$$\Delta(1) := 1 \otimes 1, \quad \epsilon(1) := 1, \quad \gamma(1) := 1, \quad 1 \in \mathcal{U}(\mathcal{P}). \quad (19)$$
Now, we consider the usual differential representation, $\rho : \mathcal{U}(\mathcal{P}) \to \text{diff}(\mathcal{A})$. The $\mathcal{U}(\mathcal{P})$-module algebra is the set $\mathcal{A} = C^\infty(\mathbb{R}^{1,3})$, i.e. all smooth functions on Minkowski space. In this case, the compatibility condition (9) of the action simply constitutes the Leibniz rule. A twisted Poincaré algebra $\mathcal{U}_t(\mathcal{P})$ is constructed from a twist element with abelian generators $P_m$ in the algebra

$$
\mathcal{F}_{PP} := \exp \left( \frac{i}{2} \theta^{mn} P_m \otimes P_n \right) \in \mathcal{U}(\mathcal{P}) \otimes \mathcal{U}(\mathcal{P}),
$$

where $\theta^{mn} \in \mathbb{R}$ is a constant with antisymmetric indices and $P_m$ are the translation generators of the Poincaré algebra. The element (20) satisfies the twist equation (10) and the counit condition (11) by virtue of the abelian property of $P_m$. The explicit form of the twisted coproduct for translations and Lorentz transformations is

$$
\Delta_t^{PP}(P_m) = \Delta(P_m),
$$

$$
\Delta_t^{PP}(M_{mn}) = \Delta(M_{mn}) - \theta^{kl} \left[ \eta_{k(m} P_{n)} \otimes P_l + P_k \otimes \eta_{l(m} P_{n)} \right],
$$

where the brackets () represent symmetrization of the indices. The antipode is not modified; i.e. $\gamma_t^{PP} = \gamma$.

For space-time coordinates $\{x^m\}$ and the representation $\rho(P_m) = i \partial_m$, we have

$$
x^m \star x^n := \mu \left( \mathcal{F}_{PP}^{-1} \triangleright \left( x^m \otimes x^n \right) \right)
= \mu \left( \exp \left( -\frac{i}{2} \theta^{kl} \rho(P_k) \otimes \rho(P_l) \right) \left( x^m \otimes x^n \right) \right)
= \mu \left( \exp \left( \frac{i}{2} \theta^{kl} \partial_k \otimes \partial_l \right) \left( x^m \otimes x^n \right) \right) = x^m x^n + \frac{i}{2} \theta^{mn}.
$$

We have thus seen that the noncommutative plane $[x^m, x^n]_* = \frac{i}{2} \theta^{mn} - \frac{i}{2} \theta^{nm} = i \theta^{mn}$ is realized by the twist element (20).

We can consider a Hopf superalgebra in almost the same manner as the Poincaré algebra above. There exist twist elements which induce a Moyal-type multiplication on the superspace.\textsuperscript{10} In fact, the $\mathcal{N} = 1$ super Poincaré algebra $\mathcal{S}\mathcal{P}$ has the structure of a Hopf algebra with a graded version of the product:

$$
(a \otimes b)(c \otimes d) = (-1)^{|b||c|} \ a c \otimes b d \quad \forall a, b, c, d \in \mathcal{S}\mathcal{P}.
$$

Here, $|b|$ represents the fermionic character of $b$, which is defined as

$$
|a| = \begin{cases} 
0 & \text{if } a \text{ is bosonic}, \\
1 & \text{if } a \text{ is fermionic}.
\end{cases}
$$

All ranks of the graded product are defined in the manner of Eq. (5), except for the overall sign, which is determined by how many exchanges of fermionic elements
are involved. When \(\mathbb{K}\) contains not only a \(c\)-number but also an anticommutative number, i.e. a Grassmann number, \(\mathbb{K}\) is no longer a field but a Grassmann number ring. In that case, for the consistency of \(\mathbb{K}\) and the Hopf superalgebra, we further require the following condition for all \(k \in \mathbb{K}\) and \(a_i \in SP\):

\[
k a_1 \otimes a_2 \otimes a_3 \cdots = (-1)^{|k||a_1|} a_1 k \otimes a_2 \otimes a_3 \cdots
= (-1)^{|k||a_2|} a_1 \otimes k a_2 \otimes a_3 \cdots
= (-1)^{|k||a_1|+|a_2|} a_1 \otimes a_2 \otimes k a_3 \cdots .
\] (26)

The definitions (19) are applied to all elements \(u \in SP\), as well as \(P\). The twist element constructed from the fermionic generators \(Q\) in the super Poincaré algebra,

\[
\{ Q_\alpha, Q_\beta \} = 0, \quad \{ Q_\alpha, \bar{Q}_\dot{\alpha} \} = 2 \sigma^m a_\dot{\alpha} P_m, \quad [ Q_\alpha, P_m ] = 0, \quad [ Q_\alpha, M_{m\dot{n}} ] = -i (\sigma_{m\dot{n}})_{\alpha\beta} Q_\beta,
\] (27)

with a constant symmetric matrix \(C^{\alpha\beta} \in \mathbb{C}\),

\[
F_{QQ} := \exp \left( -\frac{1}{2} C^{\alpha\beta} Q_\alpha \otimes Q_\beta \right),
\] (28)

satisfies Eqs. (10) and (11). Here we adopt the representation \(\rho(Q_\alpha) \theta^\beta = i\delta^\beta_\alpha\) for the fermionic coordinates \(\{ \theta^\beta \}\). The twisted multiplication is

\[
\theta^\alpha \star \theta^\beta := \mu \left( F_{QQ}^{-1} \triangleright \left( \theta^\alpha \otimes \theta^\beta \right) \right)
= \mu \left( \exp \left( \frac{1}{2} C^{\gamma\delta} \rho(Q_\gamma) \otimes \rho(Q_\delta) \right) \left( \theta^\alpha \otimes \theta^\beta \right) \right)
= \mu \left( \theta^\alpha \otimes \theta^\beta - \frac{1}{2} C^{\gamma\delta} [\rho(Q_\gamma) \theta^\alpha] \otimes [\rho(Q_\delta) \theta^\beta] \right) = \theta^\alpha \theta^\beta + \frac{1}{2} C^{\alpha\beta}. \] (29)

Therefore, we find the canonical non-anticommutativity of the superspace \(\{ \theta^\alpha, \theta^\beta \}_* = C^{\alpha\beta}\). The coproducts \(\Delta_{\gamma Q}^{QQ}\) of the generators in \(SP\) are evaluated in Ref. 10).

Next, we consider other examples. The special conformal generator \(K\) and the conformal supercharge \(S\) are both possibilities as the twist element in the superconformal algebra \(SC\). The \(K-K\) twist element \(F_{KK}\) causes very complicated noncommutativities among the superspace coordinates. Actually, the commutator of the coordinates results in an infinite series expansion with respect to the noncommutativity parameter. It is difficult to obtain the closed form of the commutator. Therefore, here we consider a twist with the \(S\)-supercharges

\[
F_{SS} := \exp \left( \frac{1}{2} C^{\alpha\beta} S_\alpha \otimes S_\beta \right), \quad C^{\alpha\beta} = C^{\beta\alpha} \in \mathbb{C},
\] (30)

\[
\rho(S_\alpha) = -i \left( x_\alpha^\dot{\beta} \theta_\beta \sigma^m_\dot{\alpha} \delta^\dot{\beta} - i \theta^2 \sigma^m_\alpha \bar{\theta} \bar{\delta} \right) \partial_m + 2 i \theta^2 \partial_\alpha + \left( x_\alpha^\dot{\beta} + 2 i \theta_\alpha \bar{\theta} \right) \bar{\delta}. \] (31)
This twist element induces the non-canonical commutation relation
\[ [x^m, x^n]_\star = C^\alpha\gamma (x^\beta \ast x^\delta \ast \theta^\beta \ast \theta^\delta) \sigma_m \beta \sigma^n \delta . \] (32)

Unlike the case of the $K-K$ twist, the $S$-supercharge twist element induces non-commutativities only at first order in the noncommutativity parameter. The higher order terms vanish because of the nilpotency of the Grassmann coordinates $\{\theta^\alpha\}$.

§3. Classification of broken and unbroken symmetries

In this section, we study the symmetry breaking on non(anti)commutative superspaces and its relation to the twisted Hopf algebra.

Modification of the superconformal symmetry on the canonical type of non-(anti)commutative superspace (2) is studied in Refs. 7) and 8). In Ref. 8), it is found that the $R$-symmetry and the dilatation symmetry are broken, but that a linear combination of these two symmetries remains unbroken as the new dilatation symmetry. To show that, the authors of Ref. 8) calculated the commutators of the generators with the star product explicitly.

Actually, from the point of view of the twisted algebra, it is obvious what symmetry is broken,\(^\ast\*) or what combination of symmetries is unbroken under the non-(anti)commutative deformation of (super)space.

The Drinfel’d twist modifies the multiplication rule in the representation space, and simultaneously changes the coproduct. This modification of the coproduct in turn changes the action on the product of the representation, and this ensures that the structure of the algebra is preserved. Thus, symmetry breaking emerges as the appearance of extra terms of the modified coproduct, which include the noncommutativity parameters. Inspecting there extra terms in the coproduct, it is clear whether any given symmetry $G$ is broken or not. For instance, in the $Q-Q$ twisted case, the coproduct corresponding to a symmetry generator $G$ becomes the following:

\[ \Delta_{\star}^{QQ}(G) = G \otimes 1 + 1 \otimes G \]
\[ - \frac{1}{2} C^{\alpha\beta} \left( Q_\alpha \otimes [Q_\beta, G]_\pm + (-1)^{|G|} [Q_\alpha, G]_\pm \otimes Q_\beta \right) \]
\[ - \frac{1}{8} C^{\gamma\delta} C^{\alpha\beta} \left( Q_\gamma Q_\alpha \otimes [Q_\delta, [Q_\beta, G]_\pm]_\pm + [Q_\gamma, [Q_\alpha, G]_\pm]_\pm \otimes Q_\delta Q_\beta \right), \] (33)

where $[A, B]_\pm$ is the (anti)commutator bracket, which gives $AB - (-1)^{|A||B|} BA$. The terms of order $O(C^3)$ vanish because $Q$ is nilpotent. From Eq. (33), it is seen that the problem of determining whether or not the symmetry is broken or not reduces to that of determining whether or not the commutator $[Q, G]_\pm$ is nonzero.

\(^\ast\) To properly express the totally (anti)symmetrized terms we should employ the by star product on the right-hand side of Eq. (32). However, the additional terms included by so doing would not change the result presented in Eq. (32).

\(^\ast\*) In this section, we use the terms “broken” and “unbroken” with regards to symmetry in the usual sense. Specifically, they do not refer to the twisted symmetry.
We focus on the $\mathcal{N} = 1$ superconformal algebra. Possible abelian twist elements\(^*\) in the superconformal algebra are listed in Table I. That table presents the twist elements with our notation, $\mathcal{F}_{XY} = \exp \left( C^{AB} \left[ X_A \otimes Y_B - (-1)^{|X||Y|} Y_B \otimes X_A \right] \right)$, and each column represents the simultaneous utilization of the indicated $\mathcal{F}$, namely the possible combinations of abelian twist elements. In fact, the product of the indicated elements, $\mathcal{F} := \mathcal{F}_{PP}\mathcal{F}_{PQ}\mathcal{F}_{QQ}$, satisfies the conditions (10) and (11) because the generators in their twist elements are in an abelian subsector.

Let us consider the unbroken symmetries in each twist element and the corresponding non(anti)commutative superspaces. Below we list the twists and the relevant commutators in the spinor representation.

- **Relevant commutators in the $P\cdot P$ twist:**
  \[
  \begin{align*}
  [P_{\alpha\dot{\alpha}}, P_{\beta\dot{\beta}}] &= 0, \\
  [P_{\alpha\dot{\alpha}}, M_{\beta\gamma}] &= -i\epsilon_{\alpha(\beta} P_{\gamma)\dot{\alpha}}, \\
  [P_{\alpha\dot{\alpha}}, Q_\beta] &= 0, \\
  [P_{\alpha\dot{\alpha}}, S_\beta] &= 2\epsilon_{\alpha\beta} \bar{Q}_{\dot{\alpha}}, \\
  [P_{\alpha\dot{\alpha}}, K_{\beta\dot{\beta}}] &= 4i \left( \epsilon_{\alpha\beta} M_{\dot{\alpha}\dot{\beta}} + \epsilon_{\dot{\alpha}\dot{\beta}} M_{\alpha\beta} - \epsilon_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} D \right), \\
  [P_{\alpha\dot{\alpha}}, D] &= iP_{\alpha\dot{\alpha}}, \\
  [P_{\alpha\dot{\alpha}}, R] &= 0.
  \end{align*}
  \]

- **Relevant (anti)commutators in the $Q\cdot Q$ twist:**
  \[
  [Q_\alpha, P_{\beta\dot{\beta}}] = 0,
  \]

\(^*\) In fact, $M_{12}$ and $P_3$ commute, but this combination is not considered here.
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Table II. Broken and unbroken symmetries.

<table>
<thead>
<tr>
<th>Twist</th>
<th>Non(anti)commutativity</th>
<th>Broken Symmetries</th>
<th>Unbroken Symmetries</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P-P$</td>
<td>$[x^m, x^n] = i\theta^{mn}$</td>
<td>$M_{a\beta}, \bar{M}<em>{a\bar{\beta}}, S</em>\alpha, \bar{S}<em>\alpha, K</em>{a\bar{a}}, D$</td>
<td>$P_{a\bar{a}}, Q_\alpha, \bar{Q}_\alpha, R$</td>
</tr>
<tr>
<td>$Q-Q$</td>
<td>${\theta^n, \theta^n} = C^{n\beta}$</td>
<td>$M_{a\beta}, Q_\alpha, S_\alpha, K_{a\bar{a}}, D, R$</td>
<td>$P_{a\bar{a}}, M_{a\bar{\beta}}, Q_\alpha, S_\alpha, D - iR$</td>
</tr>
<tr>
<td>$S-S$</td>
<td>$[x^m, x^n] \sim C_{xx\theta^\theta^\theta^\theta}$</td>
<td>$P_{a\bar{a}}, M_{a\bar{\beta}}, \bar{Q}_\alpha, D, R$</td>
<td>$M_{a\bar{\beta}}, \bar{Q}<em>\alpha, S</em>\alpha, K_{a\bar{a}}, D - iR$</td>
</tr>
<tr>
<td>$K-K$</td>
<td>$[x^m, x^n] \sim \Theta_{xxxx} + \cdots$</td>
<td>$M_{a\bar{\beta}}, \bar{M}<em>{a\beta}, Q</em>\alpha, P_{a\bar{a}}, D$</td>
<td>$S_\alpha, \bar{S}<em>\alpha, K</em>{a\bar{a}}, R$</td>
</tr>
<tr>
<td>$D-D$</td>
<td>$x^m \star x^n = e^{q_\alpha x^n} x^n$</td>
<td>$P_{a\bar{a}}, Q_\alpha, \bar{Q}<em>\alpha, S</em>\alpha, \bar{S}<em>\alpha, K</em>{a\bar{a}}$</td>
<td>$M_{a\beta}, \bar{M}_{a\bar{\beta}}, D, R$</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
[Q_\alpha, M_{\beta\gamma}] &= -i\epsilon_{\alpha(\beta Q_\gamma)}, & [Q_\alpha, \bar{M}_{\beta\gamma}] &= 0, \\
\{Q_\alpha, Q_\beta\} &= 0, & \{Q_\alpha, \bar{Q}_\alpha\} &= 2P_{a\bar{a}}, \\
\{Q_\alpha, S_\beta\} &= 4iM_{a\beta} - 2i\epsilon_{a\beta D} - 6\epsilon_{a\beta R}, & \{Q_\alpha, \bar{S}_\beta\} &= 0, \\
[Q_\alpha, K_{\beta\bar{\beta}}] &= 2\epsilon_{a\beta\bar{\beta}}\bar{S}_\beta, \\
[Q_\alpha, D] &= i\frac{1}{2}Q_\alpha, & [Q_\alpha, R] &= \frac{1}{2}Q_\alpha. \quad (35)
\end{align*}
\]

- Relevant (anti)commutators in the $S-S$ twist:

\[
\begin{align*}
[S_\alpha, Q_{\beta\bar{\beta}}] &= 2\epsilon_{\alpha\beta\bar{\beta}}\bar{Q}_\beta, \\
[S_\alpha, M_{\beta\gamma}] &= -i\epsilon_{\alpha(\beta S_\gamma)}, & [S_\alpha, \bar{M}_{\beta\gamma}] &= 0, \\
\{S_\alpha, Q_\beta\} &= 4iM_{a\beta} + 2i\epsilon_{a\beta D} + 6\epsilon_{a\beta R}, & \{S_\alpha, \bar{Q}_\beta\} &= 0, \\
\{S_\alpha, S_\beta\} &= 0, & \{S_\alpha, \bar{S}_\beta\} &= 2K_{a\bar{a}}, \\
[S_\alpha, K_{\beta\bar{\beta}}] &= 0, \\
[S_\alpha, D] &= -\frac{i}{2}S_\alpha, & [S_\alpha, R] &= -\frac{1}{2}S_\alpha. \quad (36)
\end{align*}
\]

- Relevant commutators in the $D-D$ twist:

\[
\begin{align*}
[D, P_{a\bar{a}}] &= -iP_{a\bar{a}}, & [D, M_{a\beta}] &= 0, & [D, \bar{M}_{\bar{a}\bar{\beta}}] &= 0, \\
[D, Q_\alpha] &= -i\frac{1}{2}Q_\alpha, & [D, \bar{Q}_\alpha] &= -i\frac{1}{2}\bar{Q}_\alpha, \\
[D, S_\alpha] &= i\frac{1}{2}S_\alpha, & [D, \bar{S}_\alpha] &= i\frac{1}{2}\bar{S}_\alpha, \\
[D, K_{a\bar{a}}] &= iK_{a\bar{a}}, & [D, D] &= 0, & [D, R] &= 0. \quad (37)
\end{align*}
\]

The broken and unbroken symmetries of the twist elements are given in Table II. For the $Q-Q$ and $S-S$ twists, although $D$ and $R$ respectively are broken individually, the only combination $D - iR$ survives, because both $[D, Q_\alpha] - i[R, Q_\alpha]$ and $[D, S_\alpha] - i[R, \bar{S}_\alpha]$ are zero. This agrees with the result obtained in Ref. 8).

\footnote{\indent We can replace each usual product on the right-hand sides of the commutators by the star product.}
§4. Twisted coproducts

The symmetry breaking on each non(anti)commutative space can be interpreted in terms of the extra terms in the twisted coproduct. However, in the sense of Eq. (15), the symmetries are not broken in terms of the twisted Hopf algebra, i.e. modifying the multiplication and the Leibniz rule on the original (anti)commutative space.

In this section, we investigate the twisted coproducts using abelian twist elements constructed from the generators of the superconformal algebra \( SC \). The possible such twist elements are \( P-P, K-K \) and \( D-D \) for bosonic generators and \( Q-Q \) and \( S-S \) for fermionic generators. In addition to these twist elements, there are some mixed-type twist elements, which are composed of bosonic and fermionic generators.

Because a twist element associated with bosonic generators has an infinite expansion series, naively one may guess that its twisted coproduct consists of infinitely many terms in general. For instance, the case of Eq. (20) is calculated as

\[
\Delta^P_P(G) = \Delta_0(G) + \sum_{h=1}^{\infty} \frac{1}{h!} \left[ \prod_{i=1}^{h} \frac{i}{2} \theta^{m_j n_j} \right] \times \left[ P_{m_1} \cdots P_{m_h} \otimes \text{ad}_{P_{n_1}} \cdots \text{ad}_{P_{n_h}}(G) + \text{ad}_{P_{m_1}} \cdots \text{ad}_{P_{m_h}}(G) \otimes P_{n_1} \cdots P_{n_h} \right],
\]

where \( \text{ad}_A(B) := [A, B] \) and \( G \in SC \). However, in view of Eq. (34), we obtain the adjoint nilpotency \( [P_m, [P_n, [P_k, G]]] = 0 \) for all \( G \in SP \). Hence this polynomial is only second order in the parameter \( \theta \). For the same reason, the \( K-K \) twisted coproduct has an expansion that terminates at second order in the noncommutativity parameter. In fact, for the twist element

\[
\mathcal{F}_{KK} := \exp (\Theta^{mn} K_m \otimes K_n),
\]

we have

\[
\Delta^K^K(G) = \Delta_0(G) + \Theta^{mm} (K_m \otimes [K_n, G] + [K_m, G] \otimes K_n) + \frac{1}{2} \Theta^{kl} \Theta^{mn} (K_k K_m \otimes [K_l, [K_n, G]] + [K_k, [K_m, G]] \otimes K_l K_n).
\]

Here, \( \Theta^{mn} \) is a non-zero noncommutativity parameter. The \( D-D \) twist is an exceptional case. Some coproducts have terms containing all orders of the noncommutativity parameter. Nonetheless, by virtue of the desired property of dilatation, \( [D, G] = f_G G \), with some structure constant \( f_G \in C \), the \( D-D \) twisted coproduct can be written in the following simple form:

\[
\mathcal{F}_{DD} := \exp (c D \otimes D),
\]

\[
\Delta^D_D(G) = \Delta_0(G) + \sum_{h=1}^{\infty} \frac{c^h}{h!} \left[ D^h \otimes \text{ad}_D \cdots \text{ad}_D(G) + \text{ad}_D \cdots \text{ad}_D(G) \otimes D^h \right]
\]

where \( c \) and \( f_G \) are structure constants.
\[ \Delta_0(G) + \sum_{h=1}^{\infty} \frac{(cf_G)^h}{h!} \left( D^h \otimes G + G \otimes D^h \right) = \exp (cf_G D) \otimes G + G \otimes \exp (cf_G D), \]  
\[ \text{(42)} \]

where \( c \in \mathbb{C} \) is a non-zero parameter. The \( Q-Q \) and \( S-S \) twist elements and the corresponding twisted coproduct expansion terminates at finite order in the non(anti)commutative parameter \( C \), as discussed in reference to Eq. (33). Terms of third order, \( O(C^3) \), drop out, due to the nilpotency of generators, such as \( Q_\alpha Q_\beta Q_\gamma = 0 \ (\alpha, \beta, \gamma \in \{1, 2\}) \). Hence, we can evaluate the \( S-S \) twisted coproducts as

\[ \Delta^{SS}_t(G) = \Delta_0(G) - \frac{1}{2} C^{\alpha\beta} (S_\alpha \otimes [S_\beta, G]_\pm + (-1)^{[G]} [S_\alpha, G]_\pm \otimes S_\beta) \\
- \frac{1}{8} C^{\gamma\delta} C^{\alpha\beta} (S_\gamma S_\alpha \otimes [S_\delta, [S_\beta, G]_\pm]_\pm + [S_\gamma, [S_\alpha, G]_\pm]_\pm \otimes S_\delta S_\beta). \]  
\[ \text{(43)} \]

The coproducts for the generators of \( SC \) are as follows:

\[ \Delta^{SS}_t(P_{a\dot{a}}) = \Delta_0(P_{a\dot{a}}) + \epsilon_{\alpha\beta} C^{\beta\gamma} (S_\gamma \otimes \bar{Q}_{\dot{a}} + \bar{Q}_{\dot{a}} \otimes S_\gamma), \]  
\[ \text{(44)} \]
\[ \Delta^{SS}_t(M_{a\dot{a}}) = \Delta_0(M_{a\dot{a}}) + \frac{i}{2} C^{\gamma\delta} (S_\gamma \otimes \epsilon_{\delta(\alpha} S_{\beta)} + \epsilon_{\gamma(\alpha} S_{\beta)} \otimes S_\delta), \]  
\[ \text{(45)} \]
\[ \Delta^{SS}_t(\bar{Q}_{\dot{a}}) = \Delta_0(\bar{Q}_{\dot{a}}) - 2i C^{\beta\gamma} (S_\beta \otimes M_{a\dot{a}} - M_{a\dot{a}} \otimes S_\beta) \\
+ i \epsilon_{\alpha\beta} C^{\beta\gamma} (S_\gamma \otimes D - D \otimes S_\gamma) + 3 \epsilon_{\alpha\beta} C^{\beta\gamma} (S_\gamma \otimes R - R \otimes S_\gamma) \\
- \frac{1}{4} C^{\delta\gamma} C^{\beta\gamma} [S_\delta S_\beta \otimes (\epsilon_{\eta\gamma} S_\alpha - 2 \epsilon_{\alpha(\eta} S_{\gamma)}) \\
+ (\epsilon_{\eta\gamma} S_\alpha - 2 \epsilon_{\alpha(\eta} S_{\gamma)}) \otimes S_\delta S_\beta], \]  
\[ \text{(47)} \]
\[ \Delta^{SS}_t(D) = \Delta_0(D) + \frac{i}{2} C^{\alpha\beta} S_\alpha \otimes S_\beta, \]  
\[ \text{(49)} \]
\[ \Delta^{SS}_t(R) = \Delta_0(R) + \frac{1}{2} C^{\alpha\beta} S_\alpha \otimes S_\beta, \]  
\[ \text{(50)} \]
\[ \Delta^{SS}_t(S_{\alpha}) = \Delta_0(S_{\alpha}), \]  
\[ \text{(51)} \]
\[ \Delta^{SS}_t(\bar{S}_{\dot{a}}) = \Delta_0(\bar{S}_{\dot{a}}) + C^{\alpha\beta} (K_{a\dot{a}} \otimes S_\beta - S_\beta \otimes K_{a\dot{a}}), \]  
\[ \text{(52)} \]
\[ \Delta^{SS}_t(K_{a\dot{a}}) = \Delta_0(K_{a\dot{a}}). \]  
\[ \text{(53)} \]

A bosonic-fermionic type twist element is constructed from two kinds of generators which span some abelian subsector, as listed in Table I. For example, we have \( P-Q \), \( S-\bar{Q} \), \( K-S \), and so on. Their coproducts are slightly different from the previous two.
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As a typical case, we study the $P$-$Q$ twist. Following the definition given in Ref. 10), the twist element with a Grassmann number parameter $\lambda$ is

$$F_{PQ} = \exp \left( \frac{i}{2} \lambda^{m\alpha} [P_m \otimes Q_\alpha - Q_\alpha \otimes P_m] \right).$$

(54)

The $h$-th order terms in the twisted coproduct $\Delta^P_{PQ}(G)$ for any generators $G$ of $\mathcal{SC}$ are written in the form (with the notation $\text{ad}^\pm_A(B) := [A, B]_\pm$)

$$P_{m_1} \cdots P_{m_h} \otimes \text{ad}^\pm_{Q_{\alpha_1}} \cdots \text{ad}^\pm_{Q_{\alpha_h}} (G), \quad Q_{\alpha_1} \cdots Q_{\alpha_h} \otimes \text{ad}^\pm_{P_{m_1}} \cdots \text{ad}^\pm_{P_{m_h}} (G),$$

$$P_{m_1} \cdots P_{m_k} Q_{\alpha_{k+1}} \cdots Q_{\alpha_h} \otimes \text{ad}^\pm_{Q_{\alpha_1}} \cdots \text{ad}^\pm_{Q_{\alpha_k}} \text{ad}^\pm_{P_{m_{k+1}}} \cdots \text{ad}^\pm_{P_{m_h}} (G)$$

$$\forall k \in \{1, \cdots, h-1\},$$

(55)

and the left-right reversed counterparts. Then, for the reason mentioned above, all $O(\lambda^4)$ terms vanish in the evaluation of the twisted coproduct.

§5. Summary and discussion

In this paper we have discussed the broken and unbroken superconformal symmetries on various non(anti)commutative spaces in the context of the Drinfel’d twisted Hopf algebra. The non(anti)commutative (super)spaces are realized as the representation space of the twisted Hopf algebra. The non(anti)commutativity of the superspace depends on the choice of the twist element, e.g. the $P$-$P$ twist element $F_{PP}$ for the canonical noncommutative plane $[x^m, x^n] = i\theta^{mn}$ and the $S$-$S$ twist element for the non-canonical non(anti)commutative superspace (32). A theory defined on a noncommutative space generally exhibits a violation of some part of the original (commutative space) symmetries and some part, including linear combinations of the generators in the broken symmetry, remains intact. These broken and unbroken symmetries can be systematically classified with respect to the Drinfel’d twisted Hopf algebra.

Strictly speaking, these “broken” symmetries are not broken in a certain sense, because the algebra can be recovered, provided that the corresponding coproduct is modified, as we have seen in §4.

We also considered all abelian twist elements in the superconformal algebra and classified the structure of broken and unbroken symmetries corresponding to each choice of the twist element. In addition, we showed that the $S$-$S$ twist element $F_{SS}$ provides a non-canonical non(anti)commutative structure in the superspace that differs from that of the canonical type, resulting from the $P$-$P$ or $Q$-$Q$ twist. Nevertheless, the $S$-$S$ twisted star product can be computed explicitly, and the effects of the deformation are of finite order in the noncommutativity parameter $C$.

In physical applications, one might expect that the Hopf algebra approach, which induces non-canonical non(anti)commutativity on the (super)space provides a useful tool to classify the symmetries of low energy effective field theories in superstring theory with non-constant background fields, i.e. field theories specified by commutation
relations of the type

\[ [x^m, x^n] = C^{mn}(x, \theta, \bar{\theta}) , \quad \{ \theta^\alpha, \theta^\beta \} = C^{\alpha\beta}(x, \theta, \bar{\theta}) , \tag{56} \]

which, in general, contain the non(anti)commutative structures presented in Table II.

The study carried out in this paper may provide a general method for the investigation of the symmetry structures of superconformal theories on noncommutative geometries.

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