Appendix: Extensions of the Credit Market Model

In this section, we study naïveté-based discrimination in some empirically relevant settings that fall outside our basic model, focusing on our leading application, the credit market.

A. Unobservable Beliefs

We consider a generalization of our credit market model in which firms do not know consumers' beliefs.[38] Since the credit market seems to be relatively competitive, we assume throughout that the imperfectly competitive case applies ($t < u(b^e) - b^e$). We show that our main result that naïveté-based discrimination lowers welfare survives, and we discuss exactly what kind of information lowers welfare.

There are I possible consumer beliefs, $\hat{\beta}_1$ through $\hat{\beta}_I$, where $\hat{\beta}_1 < \hat{\beta}_2 < \cdots < \hat{\beta}_I$. For each belief type $\hat{\beta}_i$, there are two possible actual levels of β, $\beta = \hat{\beta}_i$ and $\beta = \beta_i < \hat{\beta}_i$.[39] We denote the share of belief type $\hat{\beta}_i$ by s_i, and the share of naive consumers among those with beliefs $\hat{\beta}_i$ by α_i. We assume that $|\alpha_i - \alpha_j| < 1$ for any i, j.[40]

For this version of our model, we assume that in addition to b_l, r_l, d_l, firm l's contract can include capped penalties or charges (e.g.,

38. Like us, Galperti (2015) analyzes the screening of consumers according to beliefs under time inconsistency. But because he largely assumes that all consumers are sophisticated and the problem he studies—how to provide a flexible incentive to save under uncertainty—is quite different from ours—how to lend to a mix of sophisticated and partially naive consumers—the screening contracts he derives have no obvious relationship to ours. Nevertheless, some of the considerations are related. For instance, in our setting more optimistic types derive greater utility from any contract than do more pessimistic types, and in his setting less time-inconsistent types derive more benefit from savings incentives than do more time-inconsistent types.

39. At the cost of some extra notation, it is possible to extend our results to more naive types per $\hat{\beta}_i$.

40. This rules out the unrealistic scenario in which a firm cannot distinguish two belief types in a pool, yet knows that one belief type is certain to be sophisticated and the other belief type is certain to be naive.

late or over-the-limit fees) of a specific type. Namely, the firm can require an extra payment of $\Delta_l \geqslant 0$ in period 1 and impose a fine $p_l \leqslant \overline{p}$ for making the payment in period 2 instead. Accordingly, self 1 decides—on top of how much of her loan to repay in period 1—whether to pay Δ_l in period 1 or $\Delta_l + p_l$ in period 2. As we will discuss in more detail, the role of these penalties is to ensure that a consumer with a lower $\hat{\beta}$ does not take a contract intended for a consumer with a higher $\hat{\beta}$, but for many parameter values our main result holds even absent penalties.

We look for symmetric pure-strategy equilibria in the contract-offer game played between firms, assuming that each firm can offer any finite contract menu. For tractability, we assume that a firm can break consumers' indifference between contracts at will: it can assign each belief type to the contract it prefers among the ones that maximize the type's perceived utility.

Characterization of equilibrium. We first characterize the equilibrium of our model for given s_i, α_i. Readers not interested in the theoretical details can jump to our discussion of the welfare impact of naïveté-based discrimination.

We start by noting that when beliefs are observable, the possibility of imposing a penalty p does not change equilibrium social welfare.[41] The penalty has a convenient flexibility property: for any β, firm l can choose Δ_l such that a consumer with time inconsistency β is indifferent between paying and not paying a penalty of $\overline{p}$. Hence, for any $\hat{\beta}_i$, the firm can choose Δ such that naive but not sophisticated consumers pay a penalty of $\overline{p}$, thereby collecting an unexpected payment of $\overline{p}$ from naive consumers. Since lending more does not increase the penalty firms can collect, the penalty does not affect the amount firms lend in equilibrium, and hence does not affect social welfare.[42]

We illustrate the logic of our results for unobservable beliefs by discussing why the equilibrium we find is an equilibrium; Proposition 6 establishes that no other equilibrium exists. Our construction hinges on different types' equilibrium perceived utilities gross of transportation costs. We define $\hat{U}_i$ as type $\hat{\beta}_i$'s equilibrium perceived utility when beliefs are observable, and $\hat{V}_i$ as type $\hat{\beta}_i$'s equilibrium perceived utility when beliefs are unobservable, thinking of $\hat{U}$ and $\hat{V}$ as functions from $\{1, \ldots, I\}$ to $\mathbb{R}$.

41. See the proof of Proposition 6 for a formal argument.

42. Nevertheless, the possibility of imposing penalties does change the distribution of welfare by increasing the transfer from naive to sophisticated consumers.

First we make an important observation: $\hat{V}$ is increasing. If a belief type $\hat{\beta}_i$ expects to repay a given loan early and/or not pay penalties, then so does a belief type $\hat{\beta}_j > \hat{\beta}_i$. This implies that type $\hat{\beta}_j$ expects to receive at least as much utility out of any contract as does type $\hat{\beta}_i$. Hence, $\hat{V}_i > \hat{V}_j$ is incompatible with equilibrium, since type $\hat{\beta}_j$ would prefer to deviate and take the contract type $\hat{\beta}_i$ is taking.

Now we argue that if $\hat{U}_i$ is weakly increasing, then the contracts firms offer when they know beliefs are incentive compatible and hence constitute an equilibrium, even when firms do not know beliefs. Type $\hat{\beta}_i$ believes that if she takes the contract intended for type $\hat{\beta}_j < \hat{\beta}_i$, then—since with this contract both types expect to repay early and to pay no penalty—she will receive the same utility level that type $\hat{\beta}_j$ expects. But since type $\hat{\beta}_i$'s perceived utility from the contract intended for her is higher, she prefers not to deviate in this direction.

Conversely, type $\hat{\beta}_i$ is tempted to take a contract intended for type $\hat{\beta}_j > \hat{\beta}_i$, because if she can avoid interest and penalties, she obtains higher utility. But due to the above flexibility property, a firm can design the penalty in the contract intended for type $\hat{\beta}_j$ such that type $\hat{\beta}_i$ expects to pay it while type $\hat{\beta}_j$ does not, making the contract unattractive to type $\hat{\beta}_i$.

The more difficult case arises when $\hat{U}_i > \hat{U}_{i+1}$ for some i. Then, the contracts consumers get when beliefs are observable violate the constraint that $\hat{V}_{i+1} \geqslant \hat{V}_i$—thereby violating type $\hat{\beta}_{i+1}$'s incentive-compatibility constraint—and hence cannot (all) be chosen in equilibrium. Through changing the discounts d_l, however, a firm can "iron out" decreasing parts of $\hat{U}_i$ to create an increasing $\hat{V}_i$ and thereby eliminate more optimistic types' incentive to take contracts intended for more pessimistic types. If there are two types and $\hat{U}_1 > \hat{U}_2$, for instance, the firm can lower the discount to $\hat{\beta}_1$ and increase the discount to $\hat{\beta}_2$ to equalize the perceived utilities of the two types, while holding the average markup constant. Then, type $\hat{\beta}_2$ no longer prefers to take the contract intended for type $\hat{\beta}_1$.

As it happens, this set of contracts constitutes a symmetric equilibrium. First, because it maximizes consumers' average perceived utility given the average markup, a firm cannot do better by trying to attract the two types in equal numbers. Second, although type $\hat{\beta}_1$ is more profitable than type $\hat{\beta}_2$, a firm cannot disproportionately attract these profitable consumers: because type

$\hat{\beta}_2$ derives at least as much perceived utility from any contract as does type $\hat{\beta}_1$, any contract the firm offers to attract more $\hat{\beta}_1$ types also attracts more $\hat{\beta}_2$ types.

An interesting aspect of equilibrium when ironing is necessary is that some contracts are subsidized and hence may generate negative profits for a firm. Continuing with the two-type example, if $\hat{U}_1 > \hat{U}_2$ and t is sufficiently small—that is, competition is sufficiently intense—then in the symmetric equilibrium firms lose money on type $\hat{\beta}_2$. The question arises why a firm does not pull its unprofitable contract. If it did so, its type $\hat{\beta}_2$ consumers would take its other contract and generate higher losses for the firm.

The logic above generalizes to more than two possible belief types:

PROPOSITION 6 (SEPARATION ACCORDING TO BELIEFS). For any s_i, α_i, if $\overline{p}$ is sufficiently large, then any symmetric pure-strategy equilibrium is fully separating between belief types, with the borrowed amount and interest rate equal to those in the observable case for each belief type $\hat{\beta}_i$.

Although the penalties p—which, as we have explained, prevent a more pessimistic type from taking a contract intended for a more optimistic type—are somewhat special in ensuring the generality of Proposition 6, it is worth emphasizing that for many parameter combinations, the proposition holds even in an environment without penalties. First, this is the case if $\hat{U}_i$ is decreasing. Then in equilibrium $\hat{V}_i$ is constant, so a more pessimistic type does not prefer a contract intended for a more optimistic type. Second, the same holds if $\hat{U}_i$ is increasing and consumers are only slightly naive, that is, $\beta_{i+1} > \hat{\beta}_i$ for all i. Then, if type $\hat{\beta}_i$ takes the contract intended for type $\hat{\beta}_j > \hat{\beta}_i$, she expects to repay late and pay costly interest, making the contract unattractive. Third, by continuity, the addition of penalties is unnecessary if consumers are only slightly naive and the redistributions required in the ironing are sufficiently small.

The welfare effect of naïveté-based discrimination. We proceed to identify the implications of Proposition 6 for the welfare effect of naïveté-based discrimination, assuming throughout that consumers' consumption-utility function satisfies prudence. As in our basic model, we assume that with naïveté-based discrimination firms sort consumers into two pools. We denote the shares of belief type $\hat{\beta}_i$ in the two pools by s_i^1 and s_i^2, respectively, and the shares of naive consumers among those with belief type $\hat{\beta}_i$ by α_i^1 and α_i^2, respectively. For each $h = 1, 2$, we assume that—similarly to initial beliefs—$|\alpha_i^h - \alpha_j^h| < 1$ for any i, j, and that $\overline{p}$ is sufficiently high for Proposition 6 to hold both with and without discrimination. Then Proposition 6 implies that the unobservability of beliefs does not affect the equilibrium distortion faced by a consumer with a given $\hat{\beta}_i$. Hence, both with and without price discrimination the distortion is as in the observable case. Using Proposition 5, this yields:

COROLLARY 2. For any s_i^1, s_i^2, α_i^1, α_i^2, naïveté-based discrimination lowers social welfare.

Our results also allow us to characterize exactly what kind of information lowers social welfare. Information about beliefs does not affect welfare:

COROLLARY 3. For any s_i^1, s_i^2, if $\alpha_i^1 = \alpha_i^2$ for all i, then naïveté-based discrimination does not affect social welfare.

Again, since total welfare does not depend on whether firms know beliefs, it also does not depend on how much firms know about beliefs. Of course, our equilibrium characterization implies that information about beliefs can affect the distribution of welfare across consumers with different beliefs. On the other hand, any information about the likelihood that a consumer with given beliefs is naive lowers welfare:

COROLLARY 4. If $\alpha_i^1 \neq \alpha_i^2$ for some i, then naïveté-based discrimination strictly lowers social welfare.

Our results complement in an interesting way the small literature on screening with potentially naive consumers (Eliaz and Spiegler 2006, 2008, for instance), which assumes that when contracting a firm knows consumers' ex post preferences and hence also their ex post behavior, but does not know their beliefs. Corollaries 3 and 4 imply that when it comes to third-degree naïveté-based discrimination, it is information about ex post behavior given beliefs that matters, so that the unobservability of ex post behavior is crucial.

The observation that information about beliefs is welfare-neutral but information about ex post behavior given beliefs is welfare-decreasing raises the question of what information firms actually have. We are unaware of any empirical work that would help answer this question. Nevertheless, part of the same economic logic that suggests naïveté-based discrimination is likely to be going on also implies that firms are likely to have the latter, welfare-decreasing type of information. Namely, whether or not firms have information about beliefs, information about ex post behavior given beliefs is profitable for them, so they have incentives to acquire such information.

To conclude our discussion, we show that for naïveté-based discrimination to lower welfare, it is neither necessary nor sufficient for firms to learn about a consumer's time inconsistency β.

EXAMPLE 2. Suppose there are 30 consumers, $I = 2$, and $\hat{\beta}_1 = \beta_2 < \hat{\beta}_2$. There are 10 consumers with $\hat{\beta}_1$, and they are all sophisticated. There are also 10 consumers each of type $(\hat{\beta}_2, \beta_2)$ and $(\hat{\beta}_2, \hat{\beta}_2)$. Information sorts consumers into two groups in which the numbers of each type are 4,6,5 and 6,4,5, respectively.

This piece of information does not reveal anything about a consumer's time inconsistency, since in both groups the share of consumers with $\beta = \beta_2$ is $\frac{2}{3}$. Yet by Corollary 4, the information strictly lowers welfare. Intuitively, although firms do not receive information about consumers' time inconsistency, they do receive information about the proportion of naive consumers among those of belief type $\hat{\beta}_2$. Once consumers sort according to beliefs, this information leads to a reduction in welfare. This implies that for total welfare to decrease, it is not necessary for firms to receive information about time inconsistency.

Conversely, since in our model β and $\hat{\beta}$ can be correlated, information about $\hat{\beta}$ can also provide information about β. Yet by Corollary 3, such information does not affect welfare. Hence, for total welfare to decrease, it is not sufficient for firms to receive information about time inconsistency.

B. MIXED DISTORTIONS: NONLINEAR REPAYMENT COSTS

Our basic model assumes that repayment costs are linear, thereby isolating the overborrowing distortion that results from exploiting naive consumers. Now we suppose that repayment costs are nonlinear. Then, ex post distortions arise: holding the total repayment amount constant, welfare is maximized when a consumer repays the same amount in periods 1 and 2, yet with the firm's optimal interest rate, typically neither consumer does so. Furthermore, because naive and sophisticated consumers repay differently, the distortions are heterogeneous. We have not been able to fully characterize the effect of naïveté-based discrimination in this case.

Our intuition suggests, however, that the sophisticated-side repayment distortion may in reality be economically small or nonexistent because sophisticated consumers can arrange for alternative sources of funds—from their own budgets or from other sources of credit—to avoid high interest rates. To capture this possibility in a simple way, we assume that in period 1 sophisticated consumers have access to external funds at zero interest. The instantaneous disutility from repaying an amount q is $\kappa(q) = \phi q^{\gamma}$ with $\phi > 0$, $\gamma > 1$, where the power function form is to keep our analysis tractable. Also for tractability, we keep the quasi-linear form and assume that the discount d is paid in period 3 and the consumer's utility from it is linear. Hence, when taking the loan (b_l, r_l, d_l) and repaying amounts q_1 and q_2 in periods 1 and 2, self 0's utility is $u(b_l) - \kappa(q_1) - \kappa(q_2) + d_l$. For any $r_l > 0$, a sophisticated consumer repays the entire loan to the firm in period 1, but uses alternative sources of funds to set $q_1 = q_2 = \frac{b_l}{2}$. To capture the notion that it does not make sense to use the credit contract as a savings device to store resources for period 3, we assume that at the socially optimal level of borrowing b, $\kappa'(\frac{b}{2}) \geqslant 1$. A naive consumer believes in period 0 that she will do the same as a sophisticated consumer, but—having no alternative funds or procrastinating on their use—her self 1 will choose q_1 to minimize $\kappa(q_1) + \beta\kappa((1 + r_l)(b_l - q_1))$.

In this model, our results extend for a broad set of circumstances:

PROPOSITION 7 (WELFARE WITH NONLINEAR REPAYMENT COSTS). Suppose $u'''(b) > \frac{\kappa'''(\frac{b}{2})}{4}$ for all b. Then, for any α_{ns}, α_n, α_s, naïveté-based discrimination strictly raises total lending and strictly lowers social welfare.

Part of the logic of Proposition 7 is similar to that of Proposition 5. Because firms can collect unexpected interest from naive consumers, they overlend, and naïveté-based discrimination exacerbates this overlending for two reasons. First, because $u(\cdot)$ is concave and $\kappa(\cdot)$ is convex, an increase in lending to a consumer hurts social welfare more than an equal decrease in lending to a consumer raises social welfare. Second, total lending—too high to begin with—increases so long as $u'''(b) > \frac{\kappa'''(\frac{b}{2})}{4}$. To see the intuition in a simple way, suppose that $u(\cdot)$ satisfies prudence ($u''' \geqslant 0$). As in our basic model, the risk of which pool she will be allocated to increases a prudent consumer's expected marginal utility of consumption. Unlike in our basic model, however, with a nonlinear $\kappa(\cdot)$ the same risk can also increase the consumer's expected marginal cost of consumption. But because the cost of repayment is distributed over multiple periods, the shape of $u(\cdot)$ is more important in determining the amount of consumption than the shape of $\kappa(\cdot)$. Hence, borrowing is likely to increase in many situations.[43]

Beyond overlending—a homogeneous distortion—a naive-side distortion emerges because naive consumers allocate repayment between periods 1 and 2 in an ex ante suboptimal way. In addition, the two types of distortions interact: as lending increases, the ex ante cost of repaying in a suboptimal way increases, and hence the increase in overlending due to naïveté-based discrimination increases the additional distortion naive consumers face. This is especially so because lending not only increases overall, but does so more for naive consumers.

C. Proofs

Proof of Proposition 1. If firm l charges $f_l > v$, then all consumers prefer the outside option at location l to product l, so firm l has zero demand. This has two implications: (i) $f_l > v$ cannot be part of a profitable deviation; and (ii) in a symmetric pure-strategy equilibrium, the anticipated price f must satisfy $f \leqslant v$, since otherwise a firm can deviate and set $f_l = v$, $a_l = 0$, ensuring positive demand and (since $v > c$) making positive profits. We therefore assume from now on that $f_0, f_1 \leqslant v$. This implies that

43. For instance, although our model simplifies things by considering a three-period model, in a more realistic, long-horizon model both $u(\cdot)$ and $\kappa(\cdot)$ would be derived from the consumer's instantaneous utility function over consumption. In a steady state where the consumer believes that her future consumption path is approximately flat, $u'''(b) \approx \kappa'''(\frac{b}{2})$, so the condition of the proposition holds. Furthermore, given the observation that the shape of $u(\cdot)$ matters more than the shape of $\kappa(\cdot)$ because repayment is divided over multiple periods, intuition suggests that the condition required is even weaker if repayment is divided into more periods. And in the limit as the horizon becomes infinitely long, the consumer believes that she will split repayment across many periods, and therefore that her repayment cost is approximately linear, as in our quasi-linear model.

all consumers prefer product l to the outside option at l, so all consumers purchase.

When facing the anticipated prices f_0, f_1, the consumer with taste $y \in [0, 1]$ is indifferent between the two firms if $v - f_0 - ty = v - f_1 - t(1 - y)$. We consider only deviations from a symmetric pure-strategy equilibrium for which there is such an indifferent consumer; other deviations are clearly suboptimal. Firm 0's demand is therefore $y = \frac{f_1 - f_0 + t}{2t}$, and, using the fact (established in the text) that an optimizing firm charges $a(\alpha) = (k')^{-1}(\alpha)$, its profit is $\left[f_0 + \alpha a(\alpha) - k(a(\alpha)) - c\right] \frac{f_1 - f_0 + t}{2t}$. Since the profit function is strictly concave, it is sufficient to consider local deviations from a purported equilibrium. Hence, an anticipated price $f < v$ is part of an equilibrium if and only if the marginal profit of firm 0 at $f_0 = f$ when $f_1 = f$ is 0; this is equivalent to $f = c + t - [\alpha a(\alpha) - k(a(\alpha))]$. And $f = v$ is part of an equilibrium if and only if the marginal profit of firm 0 at $f_0 = v$ when $f_1 = v$ is greater than or equal to 0; this is equivalent to $c + t - [\alpha a(\alpha) - k(a(\alpha))] \geqslant v$. Hence, there exists a unique symmetric pure-strategy equilibrium, and in this equilibrium firms charge prices $f(\alpha) = \min\{c + t - [\alpha a(\alpha) - k(a(\alpha))], v\}$, $a(\alpha) = (k')^{-1}(\alpha)$.

Our analysis in the text implies that if $\frac{k'(a)}{k''(a)}$ is strictly increasing on $[a(\underline{\alpha}), a(\overline{\alpha})]$, then $DWL(\alpha)$ is strictly convex on $[\underline{\alpha}, \overline{\alpha}]$, so for α_{ns}, α_n, α_s on this interval, naïveté-based discrimination strictly lowers welfare. Conversely, if $\frac{k'(a)}{k''(a)}$ is not strictly increasing on $[a(\underline{\alpha}), a(\overline{\alpha})]$, then there exists a subinterval $(a(\underline{\underline{\alpha}}), a(\overline{\overline{\alpha}}))$ over which it is weakly decreasing. Therefore, $DWL(\alpha)$ is weakly concave on $[\underline{\underline{\alpha}}, \overline{\overline{\alpha}}]$. Thus, for $\alpha_{ns}, \alpha_n, \alpha_s$ on this interval, naïveté-based discrimination weakly raises welfare. This shows the equivalence between statements i and ii of the Proposition.

Since $DWL(\alpha)$ is strictly concave on $[\underline{\alpha}, \overline{\alpha}]$ if and only if $\frac{k'(a)}{k''(a)}$ is strictly decreasing on $[a(\underline{\alpha}), a(\overline{\alpha})]$, an analogous argument to the one above establishes the equivalence between statements i′ and ii′ of the proposition. $\square$

Proof of Proposition 2. If firm l charges $f_l > v - k(a_l)$, then all consumers prefer the outside option at location l to product l, so firm l has zero demand. This has two implications: (i) $f_l > v - k(a_l)$ cannot be part of a profitable deviation, and (ii) in a symmetric pure-strategy equilibrium, the anticipated price f must satisfy $f \leqslant v - k(a_l)$, since otherwise a firm can deviate and set $f_l = v$, $a_l = 0$, ensuring positive demand and (since $v > c$) making positive profits. We therefore assume from now on that $f_l \leqslant v - k(a_l)$ for

$l = 0, 1$. This implies that all consumers prefer product l to the outside option at l, so all consumers purchase.

We think of a firm as first solving for the optimal contract given the perceived utility $\hat{u}$ it wants to give consumers, and then choosing $\hat{u}$. For a given $\hat{u}$, the firm's problem is

$$\max_{f,a} \ \alpha(f + a) + (1 - \alpha)f - c$$

$$\text{subject to } \ v - f - k(a) = \hat{u}.$$

Solving for f from the constraint, plugging into the maximand, and differentiating with respect to a gives that in a symmetric pure-strategy equilibrium the additional price must satisfy $k'(a(\alpha)) = \alpha$. Using this fact, the same steps as in the proof of Proposition 1 can be used to show that the symmetric pure-strategy equilibrium is unique, and in the equilibrium firms charge anticipated price $f(\alpha) = \min\{c + t - \alpha a(\alpha), v - k(a(\alpha))\}$.

To analyze the welfare effects of naïveté-based discrimination, note that if firms choose $a = a(\alpha)$ in a symmetric pure-strategy equilibrium, then the welfare loss relative to first-best is $(1 - \alpha)k(a(\alpha))$. Since the welfare loss is zero for $\alpha = 0$ and $\alpha = 1$, perfect discrimination maximizes social welfare.

We will now show that if the derivative of $\frac{k'(a)}{k''(a)}$ is positive and bounded away from 0, then there is an α^* such that the function $(1 - \alpha)k(a(\alpha))$ is strictly convex on $[0, \alpha^*]$; this implies that for α_{ns}, α_n, $\alpha_s < \alpha^*$, naïveté-based discrimination lowers welfare.

The first derivative of $(1 - \alpha)k(a(\alpha))$ is

$$-k(a(\alpha)) + (1 - \alpha)k'(a(\alpha))a'(\alpha) = -k(a(\alpha)) + (1 - \alpha)\alpha a'(\alpha).$$

The second derivative is

$$(1 - 3\alpha)a'(\alpha) + (1 - \alpha)\alpha a''(\alpha).$$

Differentiating the first-order condition $k'(a(\alpha)) = \alpha$ totally with respect to α gives

$$a'(\alpha) = \frac{1}{k''(a(\alpha))}, \quad \text{and therefore } \ a''(\alpha) = -\frac{k'''(a(\alpha))}{k''(a(\alpha))^2} \cdot a'(\alpha).$$

Hence, the second derivative can be rewritten as

$$a'(\alpha)\left[(1 - 3\alpha) - (1 - \alpha)\frac{k'(a(\alpha))k'''(a(\alpha))}{k''(a(\alpha))^2}\right],$$

which is positive for sufficiently small α since $a'(\alpha) > 0$ and because the fact that the derivative of $\frac{k'(a)}{k''(a)}$ is positive and bounded away from 0 implies that

$$\frac{k'(a(\alpha))k'''(a(\alpha))}{k''(a(\alpha))^2} < 1$$

and that it is bounded away from 1. □

Proof of Proposition 3. The same argument as in the first paragraph of the proof of Proposition 1 establishes that we can restrict attention to prices for which all consumers purchase. We again first solve for a firm's optimal contract that gives perceived gross utility $\hat{u}$ to a consumer. If the consumer bears the distortionary impact, then the problem is

$$\max_{f,a} \; \alpha(f+a) + (1-\alpha)f - c$$
$$\text{subject to} \;\; v - f = \hat{u}.$$

Because increasing a increases profits without affecting the constraint, in a symmetric pure-strategy equilibrium the additional price is $a(\alpha) = a_{max}$. Using this fact, the same steps as in the proof of Proposition 1 can be used to show that the symmetric pure-strategy equilibrium is unique, and in the equilibrium firms charge anticipated price $f(\alpha) = \min\{c + t - \alpha a_{max},\, v\}$. Since $a(\alpha) = a_{max}$ is independent of α, naïveté-based discrimination has no effect on welfare.

If the firm bears the cost, then the problem is

$$\max_{f,a} \; \alpha(f+a) + (1-\alpha)f - \alpha k(a) - c$$
$$\text{subject to} \;\; v - f = \hat{u},$$

so that in a symmetric pure-strategy equilibrium the additional price $a(\alpha)$ satisfies $k'(a(\alpha)) = 1$. Using this fact, the same steps as in the proof of Proposition 1 can be used to show that the symmetric pure-strategy equilibrium is unique, and in the equilibrium firms charge anticipated price $f(\alpha) = \min\{c + t - \alpha[a(\alpha) - k(a(\alpha))],\, v\}$ with $a(\alpha) = (k')^{-1}(1)$. Since $a(\alpha)$ is independent of α, naïveté-based discrimination has no effect on welfare. □

Proof of Corollary 1. Our proofs rely on the equilibrium prices we have solved for in Propositions 1 through 3. We are in the monopolistic case if the equilibrium anticipated price makes the consumer indifferent between purchasing and not purchasing, and in the imperfectly competitive case if the consumer strictly prefers purchasing. We begin by proving statement 1 for the case of (i) homogeneous, (ii) sophisticated-side, and (iii) naive-side distortions.

Case (i). It follows from the participation constraint that $f(\alpha) = v$. Hence, the welfare of naive consumers gross of transportation costs is $-a(\alpha)$ and naive consumers in the more naive pool are worse off since $a(\alpha)$ is increasing.

Case (ii). We now have $f(\alpha) = v - k(a(\alpha))$ and the welfare of naive consumers gross of transportation costs is $v - f(\alpha) - a(\alpha) = -[a(\alpha) - k(a(\alpha))]$. Since $a - k(a)$ is concave and $a(1)$ maximizes it, the fact that $a(\alpha)$ is increasing implies that $a(\alpha) - k(a(\alpha))$ is increasing, and hence naive consumers in the more naive pool are worse off.

Case (iii). We have $f(\alpha) = v$, so the welfare of naive consumers gross of transportation costs is $-a_{max} - k(a_{max})$ if the consumers bear the exploitation cost and $-a(\alpha) = -(k')^{-1}(1)$ if the firms bear it; sophisticated consumers' welfare gross of transportation costs is zero, and since the firms' prices are independent of α so are profits. Thus, in this case information has no impact on the market outcome.

We now consider statement (ii) for the above three cases.

Case (i). Sophisticated consumers' utility is $v - f(\alpha) = v - c - t + [\alpha a(\alpha) - k(a(\alpha))]$. Because $\alpha a(\alpha) - k(a(\alpha))$ is increasing in α, sophisticated consumers allocated to the more sophisticated pool are worse off.

Case (ii). It is easy to check that the welfare of sophisticated consumers is the same as for homogeneous distortions, so the statement follows from case (i).

Case (iii). We have $f(\alpha) = c + t - \alpha a_{max}$ if naive consumers pay the distortionary impact and $f(\alpha) = c + t - \alpha[a(\alpha) - k(a(\alpha))]$ with $a(\alpha) = (k')^{-1}(1)$ if the firm pays the distortionary cost. Because in either case $f(\alpha)$ is decreasing in α, sophisticated consumers in the more sophisticated pool are worse off. □

Proof of Proposition 4. For $t < v - c$ the proof of Proposition 1 establishes that $f(\alpha) = c + t - [\alpha a(\alpha) - k(a(\alpha))]$. The equilibrium welfare of naive consumers gross of transportation cost is thus

$$v - c - t - (1 - \alpha)a(\alpha) - k(a(\alpha)).$$

Hence, perfect naïveté-based discrimination hurts naive consumers if and only if

$$(1 - \alpha)a(\alpha) + k(a(\alpha)) < k(a(1)).$$

Let $K(\alpha) = (1-\alpha)a(\alpha) + k(a(\alpha))$. Using that $k'(a(\alpha)) = \alpha$ and that $a'(\alpha) = \frac{1}{k''(a(\alpha))}$, one has

$$\frac{dK(\alpha)}{d\alpha} = -a(\alpha) + \frac{1}{k''(a(\alpha))},$$

which implies that $\frac{dK(\alpha)}{d\alpha} > 0$ if and only if $a(\alpha)k''(a(\alpha)) < 1$. Hence if $a(\alpha)k''(a(\alpha)) < 1$ for all α, $K(\alpha) < K(1)$, and thus naive consumers are hurt by perfect discrimination. This proves statement (i). Similarly, in case $a(\alpha)k''(a(\alpha)) > 1$ for all $\alpha \in (\underline{\alpha}, 1)$, one has $K(\alpha) > K(1)$ for all $\alpha \in (\underline{\alpha}, 1)$; in this case, naive consumers benefit from perfect discrimination. This proves statement (ii). $\square$

Proof of Lemma 1. Borrowers accept a firm's contract in a symmetric pure-strategy equilibrium if and only if the utility gross of transportation costs is nonnegative. This must be the case in a symmetric pure-strategy equilibrium for otherwise firm 0 could profitably deviate through offering the contract $(b^e, 0, \frac{-(u(b^e)-b^e)}{2})$; this contract gives all consumers located to the left of $\frac{1}{2}$ a greater utility than their outside option and earns positive profits per consumer accepting it. Hence, all borrowers accept the closest firm's contract in a symmetric pure-strategy equilibrium.

We next establish that the equilibrium interest rate is $r^* = \frac{1-\beta_n}{\beta_n}$ and that the amount borrowed in equilibrium, b^*, satisfies $u'(b^*) = 1 - \frac{\alpha(1-\beta_n)}{\beta_n}$. A naive consumer who has contracted with firm l delays repayment to period 2 if $\beta_n(1+r_l) \leqslant 1$, or $r_l \leqslant \frac{1-\beta_n}{\beta_n}$; and a sophisticated consumer delays if $r_l \leqslant \frac{1-\hat{\beta}}{\hat{\beta}} < \frac{1-\beta_n}{\beta_n}$. Denoting consumers' equilibrium perceived utility gross of transportation costs by $\hat{u}_l$ and writing $I_{\{\}}$ for the indicator function, firm l's contract must solve

$$\max_{b_l, d_l} \underbrace{\alpha\left(b_l + I_{\left\{r_l \leqslant \frac{1-\beta_n}{\beta_n}\right\}} r_l \cdot b_l\right) + (1-\alpha)\left(b_l + I_{\left\{r_l \leqslant \frac{1-\hat{\beta}}{\hat{\beta}}\right\}} r_l \cdot b_l\right)}_{\text{actual repayment}} - d_l - b_l$$

$$\text{subject to } \; u(b_l) - \underbrace{\left(b_l + I_{\left\{r_l \leqslant \frac{1-\hat{\beta}}{\hat{\beta}}\right\}} r_l \cdot b_l\right)}_{\text{anticipated repayment}} + d_l = \hat{u}_l.$$

An interest rate $r_l > \frac{1-\beta_n}{\beta_n}$ is suboptimal because then lowering the interest rate to $r_l' = \frac{1-\beta_n}{\beta_n}$ increases the maximand without affecting the constraint. Similarly, for any $r_l \in \left(\frac{1-\hat{\beta}}{\hat{\beta}}, \frac{1-\beta_n}{\beta_n}\right)$ raising the interest rate to $r_l' = \frac{1-\beta_n}{\beta_n}$ increases the maximand without affecting the constraint. If $r_l \leqslant \frac{1-\hat{\beta}}{\hat{\beta}}$, then changing r_l to $r_l' = \frac{1-\beta_n}{\beta_n}$ and d_l to $d_l' = d_l - r_l b_l$ increases the maximand while satisfying the constraint. Hence, the optimal interest rate satisfies $r^* = \frac{1-\beta_n}{\beta_n}$. Using this fact, firm l's contract must solve

$$\max_{b_l, d_l} \underbrace{\alpha\left(b_l + \frac{1-\beta_n}{\beta_n} \cdot b_l\right) + (1-\alpha) b_l}_{\text{actual repayment}} - d_l - b_l$$

$$\text{subject to } u(b_l) - \underbrace{b_l}_{\text{anticipated repayment}} + d_l = \hat{u}_l.$$

Solving the constraint for d_l and plugging into the maximand yields the reduced problem

$$\max_{b_l} \; u(b_l) - b_l + \alpha \cdot \frac{1-\beta_n}{\beta_n} \cdot b_l - \hat{u}_l. \tag{3}$$

Equation (3) implies that the equilibrium level of borrowing, b^*, satisfies $u'(b^*) = 1 - \alpha\frac{1-\beta_n}{\beta_n}$.

Given r^* and b^*, consumers are willing to accept such a contract as long as $u(b^*) - b^* + d^* \geqslant 0$, and because all consumers accepted a contract in a symmetric pure-strategy equilibrium, d^* must solve

$$\max_{d \geqslant -(u(b^*)-b^*)} \left(\alpha\, r^* b^* - d\right)\left[\frac{\hat{u}_1(d) - \hat{u}_2(d^*) + t}{2t}\right]. \tag{4}$$

Suppose for a moment that the constraint $d \geqslant -(u(b^*) - b^*)$ is nonbinding and let y denote the location of the borrower indifferent between firm 0 and firm 1. Using that $y = \frac{1}{2}$ and $\frac{\partial y}{\partial d_l} = \frac{1}{2t}$ in a candidate pure-strategy symmetric equilibrium simplifies the first-order condition to give

$$-\frac{1}{2} + \frac{1}{2t}\left(\alpha r^* b^* - d^*\right) = 0.$$

Rewriting yields $d^* = \alpha r^* b^* - t$. Hence, taking the constraint into account, one has $d^* = \max\{\alpha r^* b^* - t, -(u(b^*) - b^*)\}$ in such a candidate equilibrium. Because in an optimal deviation there must exist an indifferent consumer $y \in [0, 1]$ and the maximization problem above is concave, the candidate equilibrium is indeed an equilibrium. □

Proof of Lemma 2. Since $k(a) = (u(b^e) - b^e) - (u(xa) - xa)$ for $xa \geqslant b^e$, $k(\overline{a}) = k'(\overline{a}) = 0$ and $k'(a) = [1 - u'(b)]x$ for $xa \geqslant b^e$. Thus, the equilibrium condition $u'(b^*) = 1 - \frac{\alpha(1-\beta_n)}{\beta_n}$ is equivalent to $k'(a) = \alpha$. Applying the definitions of the lemma, the sophisticated consumers' utility gross of transportation costs is

$$\begin{aligned} u(b) - b + d &= u(b^e) - b^e + d + [(u(b) - b) - (u(b^e) - b^e)] \\ &= v - f - k(a), \end{aligned}$$

and naive consumers' utility gross of transportation costs

$$\begin{aligned} u(b) - b(1 + r^*) + d &= u(b^e) - b^e + d + [(u(b) - b) \\ &\quad - (u(b^e) - b^e)] - r^* b \\ &= v - f - k(a) - \left(\frac{b}{x}\right) \\ &= v - f - a - k(a), \end{aligned}$$

exactly as in our reduced-form model. □

Proof of Proposition 5. We prove that $k(\cdot)$ as defined in Lemma 2 satisfies decreasing absolute convexity on $[\frac{b^e}{x}, \infty)$; this implies that it also satisfies statement i of Proposition 1, so that by Proposition 1 naïveté-based discrimination strictly lowers welfare for any $\alpha_{ns}, \alpha_n, \alpha_s$. We have

$$\frac{k'(a)}{k''(a)} = \frac{x(1 - u'(xa))}{-x^2\, u''(xa)}.$$

The derivative of the right-hand side with respect to a is strictly positive if $[-x^2 u''(xa)]^2 + x^4 u'''(xa)[1 - u'(xa)] > 0$, and—using

$xa \geqslant b^e$ and thus $u'(xa) \leqslant 1$—this is strictly positive if $u'''(b) \geqslant 0$ for all $b \geqslant 0$. □

Proof of Lemma 3. A consumer chooses her avoidance level to minimize

$$\min_e \tilde{a}(\theta_s - e) + \kappa(e).$$

Since $\tilde{a}_{max} < \kappa'(\theta_s)$, consumers choose an avoidance level implicitly defined through $\kappa'(e) = \tilde{a}$; denote this avoidance level by $e^*(\tilde{a})$. The utility gross of transportation cost of a sophisticated consumer accepting a contract $\tilde{f}_l, \tilde{a}_l$ equals

$$v - \tilde{f}_l - [\theta_s - e^*(\tilde{a}_l)]\tilde{a}_l - \kappa(e^*(\tilde{a})) = v - f_l,$$

that of a naive consumer equals $v - f_l - a_l$, and firm l's profit per consumer it attracts equals

$$\tilde{f}_l + [\theta_s - e^*(\tilde{a}_l)]\tilde{a}_l + \alpha[\theta_n - \theta_s]\tilde{a}_l - c = f_l + \alpha a_l - c - k(a_l).$$

Because payoffs simplify to those of the reduced-form model, equilibrium must also.

Furthermore, since

$$k'(a_l) = \kappa'(e^*(\tilde{a}_l))e^{*'}(\tilde{a}_l)\frac{1}{\theta_n - \theta_s} = \frac{\kappa'(e^*(\tilde{a}_l))}{\kappa''(e^*(\tilde{a}_l))}\frac{1}{\theta_n - \theta_s},$$

$\frac{\kappa'(e)}{\kappa''(e)}$ is strictly increasing, and $e^*(\tilde{a}_l)$ is strictly increasing, the marginal cost $k'(a)$ is strictly increasing. Hence $k(a)$ is strictly convex. Finally, $k'(0) = 0$ since $e^*(0) = 0$. □

Proof of Example 1. Proposition 1 and Lemma 3 together imply that it suffices to verify that $\frac{k'(a)}{k''(a)}$ is strictly increasing. Solving for the equilibrium avoidance cost using $\kappa'(e) = \gamma\phi e^{\gamma-1} = \tilde{a}$ yields

$$e^*(\tilde{a}) = \left(\frac{\tilde{a}}{\gamma\phi}\right)^{\frac{1}{\gamma-1}}.$$

Applying the functional form of $\kappa(e)$ to $k(a) = \kappa(e^*(\frac{a}{\theta_n - \theta_s}))$ and differentiating yields

$$\frac{k'(a)}{k''(a)} = (\gamma - 1)a,$$

which is increasing in a. □

Proof of Lemma 4. Naive consumers get value $v = \int_0^1 v'g(v')dv'$ from calling and spend $f_l + a_l$ when purchasing firm l's contract, hence their utility gross of transportation cost is $v - f_l - a_l$. Sophisticated consumers call the maximal allowed number of minutes $M - a$ but no more since their utility for an extra unit is less than 1. Hence, they get utility $\int_{G^{-1}(a_l)}^1 v'g(v')dv' = v - k(a_l)$ and spend f_l when signing firm l's contract, so that their utility gross of transportation costs is $v - f_l - k(a_l)$. It is easy to verify that $k(0) = k'(0)$ and $k(\cdot)$ is three times continuously differentiable. Firm l gets an expected revenue of $f_l + \alpha a_l$ per consumer and pays cost c. Because payoffs simplify to those of the reduced form model with a sophisticated-side distortion, equilibrium must also. □

Proof of Lemma 5. Immediate. □

Proof of Proposition 6. For any set of belief types $\{i, \ldots, i'\}$, define their average perceived utility with observable beliefs as $\hat{U}_{\{i,\ldots,i'\}} = \frac{\sum_{j=i}^{i'} s_j \hat{U}_j}{\sum_{j=i}^{i'} s_j}$ and define $\hat{V}_{\{i,\ldots,i'\}}$ correspondingly for the unobservable belief case. To construct a symmetric pure-strategy equilibrium with more than two types, we define the notion of ironing we need in general:

DEFINITION 1. The function $\hat{V} : \{1, \ldots, I\} \to \mathbb{R}$ is an admissible ironing of $\hat{U} : \{1, \ldots, I\} \to \mathbb{R}$ if (i) it is weakly increasing; and (ii) for any maximal set $\{i, \ldots, i'\}$ on which $\hat{V}$ takes the same value, $\hat{V}_{\{i,\ldots,i'\}} = \hat{U}_{\{i,\ldots,i'\}}$, and $\hat{U}_{\{i,\ldots,i''\}} \geqslant \hat{U}_{\{i''+1,\ldots,i'\}}$ whenever $i \leqslant i'' < i'$.

Intuitively, an admissible ironing $\hat{V}$ is an ironing of $\hat{U}$ that keeps the average markup the same for all sets of belief types receiving the same perceived utility, and within such a set a firm cannot separate on average more profitable consumers from less profitable ones. To understand what the definition says, note that for any maximal set of types receiving a given level of perceived utility under $\hat{V}$, the firm can change its demand from these types on the margin by changing d_i, and this does not change the firm's demand from the other types. In competing for this set of types, therefore, the firms in equilibrium choose the same average markup as when beliefs are observable, so that consumers' average perceived utility is the same ($\hat{V}_{\{i,\ldots,i'\}} = \hat{U}_{\{i,\ldots,i'\}}$). In particular, if i is the only type with perceived utility $\hat{V}_i$, then we must have $\hat{V}_i = \hat{U}_i$. In addition, among the belief types with the same perceived utility under $\hat{V}$, it cannot be the case that the higher types are more profitable on average than the lower types. If higher types were more profitable, then a firm would prefer to separate and disproportionately attract them with a contract featuring slightly better terms and a penalty that puts off lower types.

With two types, there is a unique admissible ironing: that identified in the text. In particular, if $\hat{U}_2 > \hat{U}_1$, then the unique admissible ironing has $\hat{V}_i = \hat{U}_i$. If instead $\hat{V}_i$ was flat, then the higher type would be more profitable, contradicting condition (ii) of the definition. If instead $\hat{U}_2 \leqslant \hat{U}_1$, then the unique admissible ironing is flat, and in this case the higher type is less profitable. It turns out that for any number of types, there is exactly one admissible ironing:

LEMMA 6. *For any $\hat{U}$, there exists exactly one admissible ironing.*

Proof of Lemma 6. We begin by establishing that an admissible ironing exists. Our proof is constructive, and looks for the maximal sets over which consumers receive the same perceived utility, $\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_J$, defining the admissible ironing. We use an iterative algorithm. We start with $J^1 = I$, $\mathcal{I}_j^1 = \{j\}$ for $j = 1, \ldots, I$; that is, each belief type is in a singleton set.

In step m, we find the lowest j such that $\hat{V}_{\mathcal{I}_j^m} \geqslant \hat{V}_{\mathcal{I}_{j+1}^m}$. Then, we "iron out" this drop by setting $J^{m+1} = J^m - 1$, $\mathcal{I}_{j'}^{m+1} = \mathcal{I}_{j'}^m$ for $j' < j$, $\mathcal{I}_j^{m+1} = \mathcal{I}_j^m \cup \mathcal{I}_{j+1}^m$, and $\mathcal{I}_{j'}^{m+1} = \mathcal{I}_{j'+1}^m$ for $j' > j$; similarly, we set $\hat{V}_{\mathcal{I}_{j'}^{m+1}} = \hat{V}_{\mathcal{I}_{j'}^m}$ for $j' < j$, $\hat{V}_{\mathcal{I}_{j'}^{m+1}} = \hat{V}_{\mathcal{I}_{j'+1}^m}$ for $j' > j$, and $\hat{V}_{\mathcal{I}_j^{m+1}}$ as that average of $\hat{V}_{\mathcal{I}_j^m}$ and $\hat{V}_{\mathcal{I}_{j+1}^m}$ that holds the average markup constant.

Because there are a finite number of belief types, the algorithm terminates in finitely many steps. By construction, at each step each $\mathcal{I}_j^m$ satisfies condition (ii) of an admissible ironing. Again by construction, when the algorithm terminates, $\hat{V}$ is increasing. Hence, the algorithm converges to an admissible ironing.

To prove that the admissible ironing is unique, it is useful to note that for a maximal set $\{i, \ldots, i'\}$ of an admissible ironing, condition (ii) implies that for any $i \leqslant i'' \leqslant i'$,

$$\sum_{j=i}^{i''} s_j \hat{V}_j \leqslant \sum_{j=i}^{i''} s_j \hat{U}_j,$$

and that the above holds with equality if $i'' = i'$.

We now argue by contradiction that there is a unique admissible ironing. Suppose otherwise, that is, there exist two admissible ironings $\hat{V}$ and $\hat{V}'$. Clearly, condition (ii) implies that the maximal sets over which belief types get the same perceived utility must differ across these ironings. Consider the lowest set for which the belief types differ, and without loss of generality let the maximal set $\{i, \ldots, i'\}$ of the admissible ironing $\hat{V}$ be contained in the maximal set $\{i, \ldots, \tilde{i}\}$ of the admissible ironing $\hat{V}'$. Condition (ii) of an admissible ironing, as applied to maximal set $\{i, \ldots, i'\}$ of $\hat{V}$, implies that

$$\sum_{j=i}^{i'} s_j \hat{V}_j = \sum_{j=i}^{i'} s_j \hat{U}_j,$$

and as applied to higher maximal sets of $\hat{V}'$, it implies that

$$\sum_{j=i'+1}^{\tilde{i}} s_j \hat{V}_j \leqslant \sum_{j=i'+1}^{\tilde{i}} s_j \hat{U}_j. \tag{5}$$

Condition (i) of an admissible ironing implies that $\hat{V}$ is increasing, thus

$$\frac{\sum_{j=i}^{i'} s_j \hat{V}_j}{\sum_{j=i}^{i'} s_j} < \frac{\sum_{j=i'+1}^{\tilde{i}} s_j \hat{V}_j}{\sum_{j=i'+1}^{\tilde{i}} s_j}.$$

The last inequality together with inequality (5), implies that

$$\hat{U}_{\{i,\ldots,i'\}} = \hat{V}_{\{i,\ldots,i'\}} = \frac{\sum_{j=i}^{i'} s_j \hat{V}_j}{\sum_{j=i}^{i'} s_j} < \frac{\sum_{j=i'+1}^{\tilde{i}} s_j \hat{U}_j}{\sum_{j=i'+1}^{\tilde{i}} s_j} = \hat{U}_{\{i'+1,\ldots,\tilde{i}\}},$$

and hence that the set $\{i, \ldots, \tilde{i}\}$ of the admissible ironing $\hat{V}'$ violates condition (ii) of an admissible ironing, a contradiction. This completes the proof of the lemma. □

We are now ready to prove the proposition. We prove the following statement, which expands on the statement of the proposition: for any s_i, α_i, if $\overline{p}$ is sufficiently large, then any symmetric pure-strategy equilibrium is fully separating between belief types, with the borrowed amount and interest rate equal to those in the observable case for each belief type $\hat{\beta}_i$, and the discounts chosen such that $\hat{V}$ is the admissible ironing of $\hat{U}$.

We begin by establishing the analogue of Lemma 1 for the observable case in which the firm also chooses Δ_i, p_i, and in which transportation costs t are low enough such that we are in the imperfectly competitive case.

Lemma 1′: Suppose beliefs are observable and we are in the imperfectly competitive case ($t < u(b^e) - b^e$). The contract $(b_i^*, r_i^*, d_i^*, \Delta_i^*, p_i^*)$ firms offer to belief type $\hat{\beta}_i$ in a symmetric pure-strategy equilibrium satisfies $u'(b_i^*) = 1 - \alpha_i \frac{1-\beta_i}{\beta_i}$, $r_i^* = \frac{1-\beta_i}{\beta_i}$, $p_i^* = \bar{p}$, and $d_i^* - \Delta_i^* = \alpha_i(r_i^* b_i^* + \bar{p}) - t$. Furthermore, any $\Delta_i^* \in (\frac{\beta_i \bar{p}}{1-\beta_i}, \frac{\hat{\beta}_i \bar{p}}{1-\hat{\beta}_i})$ can be used in a pure-strategy symmetric equilibrium contract.

Proof of Lemma 1′. The derivation of the equilibrium contract is almost identical to that in the case of Lemma 1, and hence we only sketch the steps that differ.

We think of firm l as optimally providing a given level of perceived utility gross of transportation costs $\hat{u}_l$ to belief type $\hat{\beta}_i$, and then selecting the optimal perceived utility. By the same argument as in Lemma 1, $r_i^* = \frac{1-\beta_i}{\beta_i}$. Next, we show that the firm can set $p_i = \bar{p}$ and choose Δ_i such that naive consumers unexpectedly pay the penalty; then, by a similar argument as for the interest rate, firms do so. Naive consumers incur the fine if $\Delta_i > \beta_i(\Delta_i + \bar{p})$, while sophisticated consumers expect not to incur it if $\Delta_i < \hat{\beta}_i(\Delta_i + \bar{p})$. Hence any $\Delta_i^* \in (\frac{\beta_i \bar{p}}{1-\beta_i}, \frac{\hat{\beta}_i \bar{p}}{1-\hat{\beta}_i})$ works. Fix any such optimal Δ_i^*. Because consumers do not anticipate paying interest or the fine, their perceived utility gross of transportation costs is $\hat{u}_l = u(b_i) - b_i - \Delta_i^* + d_i$. Using this constraint, we can rewrite the firm's maximization problem as

$$\max_{b_i} \underbrace{\alpha_i \cdot \left[\frac{1-\beta_i}{\beta_i} \cdot b_i + \bar{p}\right]}_{\text{unanticipated payment}} + u(b_i) - b_i - \Delta_i^* - \hat{u}_l. \tag{6}$$

Thus, b_i^* is implicitly defined through $u_i'(b_i^*) = 1 - \alpha_i \frac{1-\beta_i}{\beta_i}$ independently of $\hat{u}_l$ and Δ_i^*.

Analogously to the proof of Lemma 1, in the imperfectly competitive case the optimal discount in a symmetric pure-strategy equilibrium solves

$$\max_{d_i} \left[\alpha_i \left(r_i^* b_i^* + \bar{p}\right) + \Delta_i^* - d_i\right] \left[\frac{\hat{u}_1(d_i) - \hat{u}_2(d^*) + t}{2t}\right].$$

Imposing on the first-order condition, that $y = \frac{1}{2}$ and $\frac{\partial y}{\partial d_i} = \frac{1}{2t}$ in a symmetric pure-strategy equilibrium and rewriting gives that $d^* - \Delta_i^* = \alpha_i(r_i^* b_i^* + \bar{p}) - t$. The candidate equilibrium contract satisfies the consumers' participation constraint if $u(b^*) - b^* + d^* - \Delta_i^* \geqslant 0$, and hence whenever this condition holds strictly, we are in the imperfectly competitive case. Substituting $d^* - \Delta_i^*$ into the participation constraint, one has

$$\begin{aligned} u(b^*) - b^* + \alpha_i(r^* b^* + \bar{p}) - t &\geqslant u(b^e) - b^e + \alpha_i(r^* b^e + \bar{p}) - t \\ &\geqslant u(b^e) - b^e - t > 0, \end{aligned}$$

where the first inequality follows from the fact that b^* solves the maximization problem (6), and the strict inequality follows from the assumption that $t < u(b^e) - b^e$. □

We proceed in five steps. Step (i) makes a preliminary observation about the equilibrium with observable belief types that is helpful in solving for the equilibrium with unobservable ones. Step (ii) establishes that the maximal perceived utility difference between belief types in the unobservable case is less than the maximal difference in the observable case. This allows us to establish that the firm can always use the penalty in the unobservable belief case to deter lower belief types from deviating and choosing contracts designed for higher belief types. Building on this, step (iii) establishes that every belief type borrows the same amount and pays the same interest rate as in the observable belief case; hence the perceived surplus is maximized also in the unobservable belief case. Step (iv) uses these facts to establish that any equilibrium is an admissible ironing that maximizes perceived surplus. Step (v) verifies that an admissible ironing that maximizes perceived surplus is indeed a symmetric equilibrium. Finally, since the admissible ironing is unique by Lemma 6, steps (iii) to (v) directly imply the proposition.

Denote the highest perceived utility of any type in the observable case by $\hat{U}_{max} = \max_i\{\hat{U}_i\}$, and denote the lowest by $\hat{U}_{min} = \min_i\{\hat{U}_i\}$.

Step (i): *If $|\alpha_i - \alpha_j| < 1$ for all $i, j \in \{1, \dots, I\}$, then there exists a $\bar{\bar{p}}$ such that for all $\bar{p} > \bar{\bar{p}}$, $\hat{U}_{max}(\bar{p}) - \hat{U}_{min}(\bar{p}) < \bar{p}$.*

The proof of Lemma 1′ implies that $\frac{d\hat{U}_i(\bar{p})}{d\bar{p}} = \alpha_i$, so the condition $|\alpha_i - \alpha_j| < 1$ ensures that for sufficiently high $\bar{p}$ the difference between any pair $\hat{U}_i(\bar{p}) - \hat{U}_j(\bar{p}) = |\alpha_i - \alpha_j|\bar{p} + \hat{U}_i(0) - \hat{U}_j(0) < \bar{p}$, and thus also that $\hat{U}_{max}(\bar{p}) - \hat{U}_{min}(\bar{p}) < \bar{p}$. We assume from here on that $\bar{p}$ is sufficiently high for this inequality to be satisfied.

Step (ii): *The equilibrium perceived utility levels with unobservable belief types satisfy $\hat{U}_{max}(\bar{p}) \geqslant \hat{V}_i \geqslant \hat{U}_{min}(\bar{p})$ for all i.*

We first show that $\hat{V}_i \leqslant \hat{U}_{max}(\bar{p})$. Suppose otherwise, that $\hat{V}_i > \hat{U}_{max}(\bar{p})$ for some i. Consider all types i that receive the highest perceived utility, and denote their set by $\tilde{\mathcal{I}}$. As we have argued in the text, $\hat{V}_i$ is weakly increasing, so if $i \in \tilde{\mathcal{I}}$ so are all higher belief types. Let $\underline{i}$ be the lowest type that receives maximal perceived utility $\hat{V}_{\tilde{\mathcal{I}}}$.

Observe that the firm must earn strictly less than t from any consumer type i in the pool $\tilde{\mathcal{I}}$; for if this was not the case, then the firm's contract in the unobservable case $(b_i, r_i, d_i, \Delta_i, p_i)$ would give a consumer of type i strictly higher utility (and hence increase the firm's demand), and at the same time earn a weakly greater profit margin than in the observable case, contradicting the fact that the contract in the observable case is profit-maximizing.

We now identify a profitable deviation for firm 0 by modifying the contracts in two steps. In the first step, for each $i \in \{1, \ldots, I\}$, we modify the contract for belief type $\hat{\beta}_i$ in the following ways (and leaving everything else unchanged):

(a) If the consumer expects to pay the penalty p_i, we set $p_i = \bar{p}$ and modify Δ_i such that she does not expect to pay the penalty, but the naive consumer pays the penalty and all lower belief types expect to pay the penalty. For this purpose,

$$\Delta_i = \bar{p} \cdot \max \left\{ \frac{\frac{\hat{\beta}_i + \hat{\beta}_{i-1}}{2}}{1 - \frac{\hat{\beta}_i + \hat{\beta}_{i-1}}{2}}, \frac{\frac{\hat{\beta}_i + \beta_i}{2}}{1 - \frac{\hat{\beta}_i + \beta_i}{2}} \right\},$$

works.

(b) If the consumer expects to pay interest, we raise r_i until she barely does not expect to pay interest, but lower belief types still expect to pay interest.

(c) We adjust d_i to keep the perceived utility of belief type $\hat{\beta}_i$ the same.

Note that this modification weakly raises profits from belief type $\hat{\beta}_i$.

We argue that with these changes, each belief type $\hat{\beta}_i$ is willing to take the contract intended for her; invoking our assumption that a firm can assign a belief type to any contract among the contracts between which the type is indifferent, the deviating firm can get consumers to take these contracts.

We start with downward deviations. For any $i' < i$, type $\hat{\beta}_i$ has the same perceived utility from the contract intended for type $\hat{\beta}_{i'}$ as does $\hat{\beta}_{i'}$ (since she expects to behave the same way); but since $\hat{V}_j$ is increasing in j, type $\hat{\beta}_i$ does not want to deviate in this direction.

Now we consider upward deviations. If type $\hat{\beta}_i$ takes the contract intended for type $\hat{\beta}_{i'} > \hat{\beta}_i$, she anticipates incurring the maximal fine and anticipates to pay weakly more interest than belief type $\hat{\beta}_{i'}$. We now argue that she will not get a higher utility from the contract intended for i' after the contract change than before the change. First, suppose i' expected to pay the fine before the contract change; then after the contract change, the discount $d_{i'}$ was reduced to reflect the fact that i' does not anticipate incurring the fine anymore, and hence for all lower types who still do anticipate incurring the fine, the contract designed for i' becomes strictly less attractive. Second, suppose i' anticipated incurring interest; then after the contract change to hold $\hat{V}_{i'}$ fixed the discount was reduced by the amount of interest that belief type i' now does not anticipate paying anymore; hence the contract does not become more attractive to lower belief types and since they (weakly) preferred their own contract over that designed for type i' prior to the contract change, they still do so after the contract change.

Take these new contracts as given. In the second step of identifying a profitable deviation, we let firm 0 reduce d_i for all belief types $i \geqslant \underline{i}$ by ϵ, so that following the deviation it offers these types a perceived utility $\hat{V}_i' = \hat{V}_i - \epsilon$, where we restrict ourselves to deviations in which $\epsilon < \hat{V}_{\underline{i}} - \hat{V}_{\underline{i}-1}$ and where we define $\hat{V}_{\underline{i}-1} = 0$ (i.e., equal to the perceived utility of the outside option gross of transportation costs) in case all types get the highest utility level in the candidate equilibrium. First, observe that when firm 0 deviates no belief type i has an incentive to choose a contract from firm 0 that was not designed for her. Since $\hat{V}_i'$ is weakly increasing, no belief type wants to choose a contract designed for a lower belief type. Furthermore, because the contracts for all types $i \geqslant \underline{i}$ now give strictly lower perceived utility to anyone accepting them, the fact that we started in a candidate equilibrium implies that no belief type $i < \underline{i}$ wants to choose a contract designed for belief types that belong to $\tilde{I}$. Finally, since the perceived utility of all contracts designed for $i \geqslant \underline{i}$ change by the same amount, no consumer $i \in \tilde{I}$ has an incentive to switch to a different contract in this set, and hence our tie-breaking assumption ensures that all consumers choose the contract designed for them.

Using the fact that $\frac{\partial y_i}{\partial d_i} = \frac{1}{2t}$ as long as all consumers keep choosing the contract designed for them and denoting the average per consumer profit of the firm in pool i by π_i, the marginal profit of changing d_i for all $i \geqslant \underline{i}$ by the same amount is

$$\sum_{i=\underline{i}}^{I} s_i \left[-\frac{1}{2} + \frac{\partial y_i}{\partial d_i} \pi_i \right] = \sum_{i=\underline{i}}^{I} s_i \left[-\frac{1}{2} + \frac{1}{2t} \pi_i \right] < \sum_{i=\underline{i}}^{I} s_i \left[-\frac{1}{2} + \frac{1}{2t} t \right] = 0,$$

where we have used that in a symmetric pure-strategy equilibrium firm l has market share $\frac{1}{2}$, and the inequality follows from the fact that the average per consumer profit for all consumer belief types $i \geqslant \underline{i}$ is less than t. Thus, firm 0's profits increase if it slightly lowers d_i for all $i \geqslant \underline{i}$ by the same ϵ, a contradiction. We conclude that $\hat{V}_i \leqslant \hat{U}_{max}(\bar{p})$.

We next show that $\hat{V}_i \geqslant \hat{U}_{min}(\bar{p})$ for all i. Suppose otherwise, that is, $\hat{V}_i < \hat{U}_{min}(\bar{p})$ for some type i. Since $\hat{V}_i$ is weakly increasing, there exists a set of belief types $\{1, \cdots, \bar{i}\}$ that gets perceived utility $\hat{V}_i < \hat{U}_{min}(\bar{p})$. Consider firm 1 deviating and replacing any contract signed by a belief type $i \leqslant \bar{i}$ with a contract in which $b_i' = b_i^*$, $r_i' = r_i^*$ and

$$\Delta_i' = \bar{p} \cdot \max \left\{ \frac{\frac{\hat{\beta}_i + \hat{\beta}_{i-1}}{2}}{1 - \frac{\hat{\beta}_i + \hat{\beta}_{i-1}}{2}}, \frac{\frac{\hat{\beta}_i + \beta_i}{2}}{1 - \frac{\hat{\beta}_i + \beta_i}{2}} \right\},$$

$p_i' = \bar{p}$, and d_i' is chosen such that the perceived utility of belief type i is $\hat{V}_i(b_i', r_i', d_i', \Delta_i', p_i') = \hat{V}_i + \epsilon$, where $\epsilon < \min\{\hat{U}_{min}(\bar{p}) - \hat{V}_{\bar{i}}, \hat{V}_{\bar{i}+1} - \hat{V}_{\bar{i}}\}$. First, note that no belief type has an incentive to select a contract other than that designed for herself, and thus our tie-breaking rule ensures that all belief types choose the contract designed for themselves. To see this, we only need to check that no belief type $i' > \bar{i}$ now prefers to choose a contract intended for a type $i \leqslant \bar{i}$; the change clearly introduces no other new incentives to deviate. If type i' takes the contract intended for type i, she expects to behave the same way as type i' expects to behave, so she expects the same perceived utility. But her perceived utility from her own contract is higher. Second, note that $\Delta_i' \in (\frac{\bar{p}\beta_i}{1-\beta_i}, \frac{\bar{p}\hat{\beta}_i}{1-\hat{\beta}_i})$, and hence $b_i', r_i', \Delta_i', p_i'$ are optimal choices in the observable case. For this particular optimal Δ_i', however, the firm must be offering a lower discount d_i' than in the observable case, because the consumers receive a lower perceived utility gross of transportation costs than in the observable case (since $\hat{V}_i + \epsilon < \hat{U}_{min}(\bar{p}) \leqslant \hat{U}(\bar{p})$). Thus the per consumer profit π_i is strictly greater than t for all these belief types $i \leqslant \bar{i}$. But then the marginal profit of slightly increasing the d_i for all consumers in the pool is

$$(7) \qquad \sum_{i=1}^{\bar{i}} s_i \left[-\frac{1}{2} + \frac{\partial y_i}{\partial d_i} \pi_i \right] = -\frac{1}{2} \left(\sum_{i=1}^{\bar{i}} s_i \right) + \frac{1}{2t} \left(\sum_{i=1}^{\bar{i}} s_i \pi_i \right)$$
$$> \left(\sum_{i=1}^{\bar{i}} s_i \right) \left[-\frac{1}{2} + \frac{1}{2t} t \right] = 0,$$

a contradiction. We conclude that $\hat{U}_{max}(\bar{p}) \geqslant \hat{V}_i \geqslant \hat{U}_{min}(\bar{p})$ for all i.

Step (iii): *For any $\bar{p} > \hat{U}_{max}(\bar{p}) - \hat{U}_{min}(\bar{p})$, each belief type i's contract has $b_i = b_i^*, r_i = r_i^*, p_i = \bar{p}$, and $\Delta_i \in [\frac{\beta_i}{1-\beta_i}, \frac{\hat{\beta}_i}{1-\hat{\beta}_i}]$.*

In other words, each belief type receives a contract that can differ from an optimal contract with observable beliefs only in the discount. Suppose not. Consider the candidate equilibrium $\hat{V}_i$, and note that $\hat{V}_i$ is (weakly) increasing. Now consider firm 1 deviating and offering contracts to each belief type i such that $b_i' = b_i^*, r_i' = r_i^*$ are set as in the observable case,

$$\Delta_i' = \bar{p} \cdot \max \left\{ \frac{\frac{\hat{\beta}_i + \hat{\beta}_{i-1}}{2}}{1 - \frac{\hat{\beta}_i + \hat{\beta}_{i-1}}{2}}, \frac{\frac{\hat{\beta}_i + \beta_i}{2}}{1 - \frac{\hat{\beta}_i + \beta_i}{2}} \right\},$$

$p_i' = \bar{p}$, and d_i' is chosen such that the perceived utility of belief type i is $\hat{V}_i(b_i', r_i', d_i', \Delta_i', p_i') = \hat{V}_i$. First, observe that since $\hat{V}_i$ is weakly increasing and no contract contains a repayment option that yields higher long-run utility than $\hat{V}_i$, no belief type i has an incentive to select a contract designed for a lower type. Second, because each belief type realizes that it will incur the penalty $\bar{p}$ when choosing the contract designed for a higher type, no belief type will want to do so, because the difference in perceived utilities between any belief types $i, \hat{i}$ is less than $\hat{U}_{max}(\bar{p}) - \hat{U}_{min}(\bar{p}) < \bar{p}$. Thus, our tie-breaking rule ensures that each belief type still chooses the contract designed for herself, and firm 1's market share does not change. Firm 1's profits from each belief type, however, increases; hence this is a profitable deviation, completing the contradiction.

Step (iv): *In a symmetric pure-strategy equilibrium, $\hat{V}$ is an admissible ironing of $\hat{U}$.*

We have already established that the equilibrium contracts differ from equilibrium contracts in the observable case at most in d_i. We next argue that for any maximal set $\{i, \ldots, i'\}$ on which $\hat{V}$ takes the same value, $\hat{V}_{\{i,\ldots,i'\}} = \hat{U}_{\{i,\ldots,i'\}}$. Suppose not. If $\hat{V}_{\{i,\ldots,i'\}} > \hat{U}_{\{i,\ldots,i'\}}$ then consumers get a higher perceived utility level than in the observable case, and hence—since d_i is simply a transfer between consumers and firms—the firms' profits must be lower. Hence, the firms earn less than t on average across all belief types in the set $\{i, \ldots, i'\}$. Now consider a firm that lowers all d_i for all belief types in the set. As long as all belief types still choose the contract designed for them, this change has a positive marginal profit at the symmetric equilibrium. Furthermore, when setting

$$\Delta_i' = \bar{p} \cdot \max\left\{ \frac{\frac{\hat{\beta}_i + \hat{\beta}_{i-1}}{2}}{1 - \frac{\hat{\beta}_i + \hat{\beta}_{i-1}}{2}}, \frac{\frac{\hat{\beta}_i + \beta_i}{2}}{1 - \frac{\hat{\beta}_i + \beta_i}{2}} \right\}$$

and $p_i' = \bar{p}$, no consumer has an incentive to chose a contract designed for a higher type. And as long as the reduction in d_i is small enough such that the perceived utility $\hat{V}_{\{i,\ldots,i'\}} > \hat{V}_{i-1}$, no type has an incentive to choose a lower type's contract. Thus, in this case there is a profitable deviation.

Analogously, if $\hat{V}_{\{i,\ldots,i'\}} < \hat{U}_{\{i,\ldots,i'\}}$ the firms earn more than t on average across all belief types in the maximal set $\{i, \ldots, i'\}$ when choosing b_i, r_i, as in the observable case and setting $p_i = \overline{p}$ and $\Delta_i = \Delta_i'$. Now consider increasing the discount d_i for all consumers in the set $\{i, \ldots, i'\}$ in such a way that $\hat{V}_{\{i,\ldots,i'\}} < \hat{V}_{i+1}$; observe that the marginal profit of doing so at the symmetric equilibrium is positive. We conclude that a necessary condition for a symmetric pure-strategy equilibrium is that $\hat{V}_{\{i,\ldots,i'\}} = \hat{U}_{\{i,\ldots,i'\}}$.

To show that a symmetric pure-strategy equilibrium must be an admissible ironing, we are left to argue that for any maximal set for which $\hat{V}_{\{i,\ldots,i'\}} = \hat{U}_{\{i,\ldots,i'\}}$, $\hat{U}_{\{i,\ldots,i''\}} \geqslant \hat{U}_{\{i''+1,\ldots,i'\}}$ for all $i \leqslant i'' < i'$. Suppose otherwise, that is, there exists some $i'' \in \{i, \cdots, i'-1\}$ such that $\hat{U}_{\{i,\ldots,i''\}} < \hat{U}_{\{i''+1,\ldots,i'\}}$. Note that if both firms would choose d_i's that induce the perceived utility level $\hat{U}_{\{i''+1,\ldots,i'\}}$ for the set $\{i''+1, \ldots, i'\}$ instead, firms would earn on average t per consumer.

Hence, they must earn more than t per consumer from belief types in the set $\{i'' + 1, \ldots, i'\}$. Now consider firm 0 deviating and offering to all consumers in the set $\{i'' + 1, \ldots, i'\}$ a modified contract in which $\Delta_i = \Delta'_i$ and d_i equals the former discount plus $\Delta'_i - \Delta_i + \epsilon$ for some small $\epsilon > 0$; no lower type will choose this contract and if ϵ is small enough so that $\hat{U}_{\{i''+1,\ldots,i'\}} < \hat{V}_{i'+1}$ after the modification, no higher belief type will choose a contract designed for belief types in $\{i'' + 1, \ldots, i'\}$. But since the per consumer profit in this pool is greater than t, such a deviation raises profits of firm 0 by a calculation analogous to that in equation (7), a contradiction.

Step (v): *The contracts identified in steps (iii) and (iv) constitute an equilibrium.*

Consider possible deviations by firm 0. Note that any deviation induces a profile $\hat{V}'$ of perceived utilities, and that this profile must be weakly increasing since for any contract, a higher belief type gets a higher utility than a lower one. We enlarge firm 0's set of possible deviations by allowing it to observe belief types (and hence condition its contract offer on the observable type) with the restriction that any profile of perceived utilities it offers must be weakly increasing. (Intuitively, this amounts to ignoring the constraint that a lower belief type may want to choose a contract designed for a higher belief type.) Clearly, this is a larger class of feasible deviations, and as we will establish that there is no profitable deviation in this larger class, there is also no profitable deviation in the original game.

When belief types are observable, it follows from the proof for the observable case that the firm will offer a contract in which $(b_i, r_i, \Delta_i, p_i)$ are chosen such that they are optimal in the observable case. For each belief type fix these parameters, selecting an optimal $\Delta_i \in [\frac{\bar{p}\beta_i}{1-\beta_i}, \frac{\bar{p}\hat{\beta}_i}{1-\hat{\beta}_i}]$. From now on we only need to consider deviations in which the firm changes d_i for some belief type.

Now consider any maximal set $\{i, \ldots, i'\}$ in which the admissible ironing induces a constant perceived utility level. We first argue that it is optimal for firm 0 to induce the same utility level in any best response over such a set. Denote the profit firm 0 earns from selling a contract to belief type i by $\pi_i(d_i)$. Hence, the profits from selling to the set $\{i, \ldots, i'\}$ of belief types is

$$\sum_{j=i}^{i'} s_j \pi_j(d_j) y_j(d_j),$$

where $y_j(d_j)$ denotes the belief type j's marginal consumer willing to buy from firm 0.

Since in an admissible ironing, $\hat{U}_{\{i,\ldots,i''\}} \geqslant \hat{U}_{\{i''+1,\ldots,i'\}}$ for all $i \leqslant i'' < i'$, we argue next that firm 0 earns higher average per consumer profits from selling to belief types $\{i, \ldots, i''\}$ than from selling to belief types $\{i'' + 1, \ldots, i'\}$ for any weakly increasing $\hat{V}'$; by definition, this is true when firm 0 offers the same perceived utility level to all consumers and charges on average t (which it charges each belief type in the observable beliefs case). Since utility is transferable, it also holds for any other average amount the firm charges whenever the perceived utility level for all consumers in $\{i, \ldots, i'\}$ is the same. When the perceived utility is increasing, relative to a constant perceived utility scenario firm 0 offers greater discounts to higher than to lower types. Hence, the average per consumer profits satisfy

$$\frac{\sum_{j=i}^{i''} s_j \pi_j(d_j)}{\sum_{j=i}^{i''} s_j} \geqslant \frac{\sum_{j=i''+1}^{i'} s_j \pi_j(d_j)}{\sum_{j=i''+1}^{i'} s_j}, \tag{8}$$

with the inequality being strict in case the perceived utility level is not constant.

Now suppose for the sake of contradiction that there exists some $i'' \in \{i, \ldots, i' - 1\}$ for which $\hat{V}'_{i''} < \hat{V}'_{i''+1}$. Then for it to be unprofitable to lower the discount for all belief types in the set $\{i'' + 1, \ldots, i'\}$, it must be that

$$\frac{1}{2t} \sum_{j=i''+1}^{i'} s_j \pi_j(d_j) - \sum_{j=i''+1}^{i'} s_j y_j(d_j) \geqslant 0, \tag{9}$$

whereas for it to be unprofitable to raise the discount for all belief types in the set $\{i, \ldots, i''\}$ it must be that

$$\frac{1}{2t} \sum_{j=i}^{i''} s_j \pi_j(d_j) - \sum_{j=i}^{i''} s_j y_j(d_j) \leqslant 0. \tag{10}$$

Dividing equation (9) by $\sum_{j=i''+1}^{i'} s_j$ and equation (10) by $\sum_{j=i}^{i''} s_j$, and using that equation (8) holds with a strict inequality, this implies that

$$\frac{\sum_{j=i''+1}^{i'} s_j y_j(d_j)}{\sum_{j=i''+1}^{i'} s_j} < \frac{\sum_{j=i}^{i''} s_j y_j(d_j)}{\sum_{j=i}^{i''} s_j}.$$

But because firm 1 offers all belief types the same perceived utility while firm 0 offers strictly higher utility to the belief types in the set $\{i'' + 1, \ldots, i'\}$ than to those in $\{i, \ldots, i''\}$, the indifferent consumer in the former set is located at a higher point in the unit interval than in the latter one, contradicting the last inequality. We conclude that firm 0 offers all consumers that get the same perceived utility in the admissible ironing the same perceived utility in an optimal deviation.

To conclude the proof of this step, we show that for any maximal set $\{i, \ldots, i'\}$ on which $\hat{V}_j$ is constant, firm 0's best response is to set $\hat{V}'_j = \hat{V}_j$. To do so, we relax the constraint that $\hat{V}'$ is increasing by allowing any response such that if $\hat{V}_j$ is constant on a set $\{i, \ldots, i'\}$, then so is $\hat{V}'_j$; that is, we do not require that $\hat{V}'_j$ be increasing between pools. We show that in this larger class of possible strategies, the best response is to offer the symmetric equilibrium contract. To see this, note that for each belief type $\frac{\partial y_i}{\partial d_i} = \frac{1}{2t}$ in the case that y_i is interior and 0 otherwise. Furthermore, as both the discount and the transportation cost are additively separable and firm 1 offers the same perceived utility to all consumers in the pool, we are only considering choices of discounts that induce the same $y_i = y$ for all belief types in the maximal set. Clearly, $y = 0$ reduces profits and whenever $y = 1$ the firm does not want to increase the discount to the consumers in this set. Think of 0 as choosing a location for the indifferent consumers y, and denote the corresponding discount to belief type j as $d_j(y)$. Firm 0's problem is thus to select y to maximize

$$\sum_{j=i}^{i'} s_j \pi_j(d_j(y))\, y,$$

giving rise to the first-order condition

$$-\sum_{j=i}^{i'} s_j (2t) y + \sum_{j=i}^{i'} s_j \pi_j(d_j(y)) = 0.$$

Since in our candidate equilibrium $y = \frac{1}{2}$ and $\sum_{j=i}^{i'} s_j \pi_j(d_j(y)) = t$, it satisfies the first-order condition. Because furthermore $\pi_j(d_j(y))$ is decreasing in y, the left-hand side is decreasing in y and thus the discounts offered in an admissible ironing are the unique ones to solve firm 0's problem. We conclude that playing the symmetric equilibrium is the unique best reply. Hence, up to the possible selection of an optimal $\Delta_i \in [\frac{\bar{p}\beta_i}{1-\beta_i}, \frac{\bar{p}\hat{\beta}_i}{1-\hat{\beta}_i}]$ for each belief type and the induced necessary change in the discount to hold $\hat{V}_i$ constant, the best reply is unique.

To conclude, we note that there exists a unique symmetric pure-strategy equilibrium up to inconsequential variation in $\Delta_i \in [\frac{\bar{p}\beta_i}{1-\beta_i}, \frac{\bar{p}\hat{\beta}_i}{1-\hat{\beta}_i}]$ and the corresponding changes in d_i to hold $\Delta_i - d_i$ fixed, and that this equilibrium satisfies the properties stated in the proposition. We argued in step (iii) that for each belief type i the borrowed amount, the interest rate and the penalty are set as in the observable case, and that Δ_i is selected among those that are optimal when belief types are observable. Hence, a symmetric equilibrium is fully separating between belief types, with the borrowed amount, penalties, and interest rates as in the observable case. Furthermore, for any given optimal selection of $\Delta_1, \ldots, \Delta_I$, the discounts are chosen such that $\hat{V}$ is the admissible ironing of $\hat{U}$ by step (iv). Conversely, by step (v), an admissible ironing in which $(b_i, r_i, p_i, \Delta_i)$ are chosen such that they are optimal with observable beliefs is a symmetric pure-strategy equilibrium. $\square$

Proof of Proposition 7. We begin by solving for a firm's optimal interest rate r. Since r does not affect a consumer's perceived repayment cost and hence willingness to accept the contract, the firm chooses r to maximize ex post repayment revenue. Naive consumers who accepted a contract solve

$$\min_{q_1} \phi q_1^{\gamma} + \beta_n \phi [(1+r)(b - q_1)]^{\gamma}.$$

Solving this for q_1 yields

$$q_1 = \underbrace{\left[\frac{\beta_n^{\frac{1}{\gamma-1}} (1+r)^{\frac{\gamma}{\gamma-1}}}{1 + \beta_n^{\frac{1}{\gamma-1}} (1+r)^{\frac{\gamma}{\gamma-1}}} \right]}_{\equiv \eta(r)} b.$$

Hence, the firm chooses r to solve

$$\max_r \ \eta(r)b + (1-\eta(r))(1+r)b = b \cdot \max_r[\eta(r) + (1-\eta(r))(1+r)],$$

which yields a solution that does not depend on b. Denote naive consumers' total repayment amount when the firm optimally chooses r by zb; we know $z > 1$.

Now if a sophisticated consumer borrows b, she repays $\frac{b}{2}$ in both periods 1 and 2, yielding total repayment costs of the form $\Lambda_s(b) = \lambda_s b^\gamma$. Furthermore, the foregoing implies that naive consumers' total repayment cost is also of the form $\Lambda_n(b) = \lambda_n b^\gamma$. We show that $\lambda_n > \lambda_s$. Naive consumers' total repayment cost is

$$\begin{aligned} \phi[\eta(r)b]^\gamma + \phi[(1-\eta(r))(1+r)b]^\gamma &= \phi b^\gamma [\eta(r)^\gamma + ((1-\eta(r))(1+r))^\gamma] \\ &> \phi b^\gamma [\eta(r)^\gamma + (1-\eta(r))^\gamma] \geqslant \phi b^\gamma 2\left(\frac{1}{2}\right)^\gamma, \end{aligned}$$

where the first inequality follows from $r > 0$ and the second follows from the fact that the term in square brackets is minimized at $\eta(r) = \frac{1}{2}$.

Again thinking of the firm's problem as optimally choosing a contract such that consumers' perceived utility gross of transportation costs is $\hat{u}_l$, the firm solves

$$\max_{b,d} \ (1-\alpha)b + \alpha z b - b - d$$
$$\text{s.t. } u(b) - \Lambda_s(b) + d = \hat{u}_l$$

Solving for d in the constraint and substituting into the maximand gives

$$\max_b u(b) + \alpha(z-1)b - \Lambda_s(b) - \hat{u}_l,$$

yielding the first-order condition

$$u'(b(\alpha)) = \Lambda_s'(b(\alpha)) - \alpha(z-1). \tag{11}$$

We next establish that $b(\alpha)$ is strictly convex in α, so that naïveté-based discrimination strictly raises total lending. Differentiating equation (11) totally with respect to α and solving for the derivative of $b(\alpha)$ gives

$$b'(\alpha) = \frac{z-1}{\frac{\kappa''(\frac{b(\alpha)}{2})}{2} - u''(b(\alpha))}.$$

This derivative is strictly increasing in α since $u'''(b) > \frac{\kappa'''(\frac{b}{2})}{4}$ for all b. Hence information about α increases overall borrowing.

When firms choose the interest rate above and offer b and d, social welfare gross of transportation costs is

$$\begin{aligned}
&(1-\alpha)b + \alpha z b - b - d + (1-\alpha)\big(u(b) - \Lambda_s(b) + d\big) \\
&+ \alpha\big(u(b) - \Lambda_n(b) + d\big) \\
= {}& u(b) + \alpha(z-1)b - \Lambda_s(b) - \alpha(\Lambda_n(b) - \Lambda_s(b)).
\end{aligned}$$

We show that in equilibrium, the above is strictly concave in α. Differentiating with respect to α and using equation (11), the first derivative is

(12)

$$(z-1)b(\alpha) - (\Lambda_n(b(\alpha)) - \Lambda_s(b(\alpha))) - \alpha(\Lambda_n'(b(\alpha)) - \Lambda_s'(b(\alpha)))b'(\alpha).$$

We first observe that $\Lambda_n'(b(\alpha)) - \Lambda_s'(b(\alpha)) > (z-1)$, which together with the fact that $b'(\alpha) > 0$ implies that $(z-1)b(\alpha) - (\Lambda_n(b(\alpha)) - \Lambda_s(b(\alpha)))$ is decreasing in α. One has,

$$\begin{aligned}
\Lambda_n'(b(\alpha)) - \Lambda_s'(b(\alpha)) &= \gamma\phi b(\alpha)^{\gamma-1}\Big\{[\eta(r)^\gamma + ((1-\eta(r)) \\
&\quad \times (1+r))^\gamma] - 2\left(\frac{1}{2}\right)^\gamma\Big\} \\
&\geqslant \gamma\phi b(\alpha)^{\gamma-1}\left\{2\left(\frac{z}{2}\right)^\gamma - 2\left(\frac{1}{2}\right)^\gamma\right\} \\
&\geqslant \gamma\phi b(\alpha)^{\gamma-1}\left(\frac{1}{2}\right)^{\gamma-1}\{z^\gamma - 1\} \\
&= \kappa'\left(\frac{b(\alpha)}{2}\right)\{z^\gamma - 1\} \geqslant \{z^\gamma - 1\} > z - 1,
\end{aligned}$$

where the first inequality follows from the fact that $z = \eta(r) + (1-\eta(r))(1+r)$ and $\gamma > 1$, so that $[\eta(r)^\gamma + ((1-\eta(r))(1+r))^\gamma] \geqslant z^\gamma 2^{1-\gamma}$; the final weak inequality follows from the assumption that $\kappa'\left(\frac{b^e}{2}\right) \geqslant 1$ and the fact that $\kappa'\left(\frac{b^e}{2}\right) = \kappa'\left(\frac{b(0)}{2}\right)$ by equation (11).

Next note that the term $\alpha(\Lambda_n'(b(\alpha)) - \Lambda_s'(b(\alpha)))b'(\alpha)$ is strictly increasing in α since all three terms are. Hence, the first derivative of social welfare with respect to α is decreasing, and thus social welfare is concave in α. Thus information on α lowers social welfare. $\square$