Internet Appendix for "Stress Testing and Bank Lending"

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IA.1 Proof of Proposition 10

For $B \leq \underline{B}$, consider, first, the case if the bank originates a risky loan in the first period. The equilibria in which the bank recapitalizes in the first period if and only if it fails the stress test are identical to those characterized in Propositions 4–7. These are the reputation-building equilibrium, the no-reputation-building equilibrium, and the self-fulfilling reputation-building equilibrium.

Next, consider an equilibrium in which the bank does not recapitalize in the first period, regardless of the stress test results. We examine this by looking at the bank's investment decision in the second period contingent on the stress test result:

- $\lambda^p = \lambda^f$. That is, the bank's investment decision in the second period is the same after passing or failing the stress test in the first period. In this equilibrium, the regulator finds it optimal to pass the bank with certainty. This is the always-pass equilibrium.

- $\lambda^p > \lambda^f$. That is, the bank originates a risky loan in the second period with strictly higher probability if it passes the stress test in the first period than if it fails the stress test in the first period. In this equilibrium, the type-τ regulator's stress-testing strategy is to pass the bank if and only if:

$$\xi_\tau + \delta(\lambda^p - \lambda^f) \left[U_\tau^R(\theta_{2,\tau}^*) - U^0 \right]. \tag{IA.1}$$

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The first term reflects the direct benefit of passing the bank to save on the cost ξ_τ. There is no other cost or benefit because the bank does not recapitalize in the first period, regardless of the stress test results. The second term reflects the reputation-building incentives of the regulator. Notice that the regulator's stress-testing strategy does not depend on the quality θ_1 of the bank's risky loan. Let us denote by $\pi_\tau \in [0,1]$ the probability with which the type-τ regulator passes the bank.

If the bank passes and fails the stress test in the first period with strictly positive probabilities, then this is an equilibrium if the bank's investment decision in the second period is optimal—i.e., $\tilde{z}_2^p(\pi_\ell, \pi_h; z_1) \leq z^* \leq \tilde{z}_2^f(\pi_\ell, \pi_h; z_1)$, where

$$\tilde{z}_2^p(\pi_\ell, \pi_h; z_1) = \frac{\pi_\ell z_1}{\pi_\ell z_1 + \pi_h(1-z_1)}, \tag{IA.2}$$

$$\tilde{z}_2^f(\pi_\ell, \pi_h; z_1) = \frac{(1-\pi_\ell) z_1}{(1-\pi_\ell) z_1 + (1-\pi_h)(1-z_1)}. \tag{IA.3}$$

Notice that $\tilde{z}_2^p(\pi_\ell, \pi_h; z_1) \leq z^* \leq \tilde{z}_2^f(\pi_\ell, \pi_h; z_1)$ implies that $\pi_h \geq \pi_\ell$, which, in turn, implies that (A.12) holds. We show below that such an equilibrium exists for all $\delta \geq \frac{\xi_\ell}{U^0 - U_\ell^R(\theta_{2,\ell}^*)} > 0$, where the last inequality follows because $\underline{B} < B_\ell$. This is the money-burning equilibrium. Consider the following scenarios for the regulator's stress-testing strategy.

- Expression (A.12) holds with strict inequalities in both instances. This implies that $\pi_h = 1$ and $\pi_\ell = 0$. Therefore, we have that $\tilde{z}_2^p(0,1;z_1) = 0 < z^* < 1 = \tilde{z}_2^f(0,1;z_1)$, and, thus, $\lambda^p = 1$ and $\lambda^f = 0$. This is an equilibrium if and only if:

$$\xi_h + \delta\left[U_h^R(\theta_{2,h}^*) - U^0\right] > 0 > \xi_\ell + \delta\left[U_\ell^R(\theta_{2,\ell}^*) - U^0\right]. \tag{IA.4}$$

  That is, this is an equilibrium if and only if $\delta \in \left[\frac{\xi_\ell}{U^0 - U_\ell^R(\theta_{2,\ell}^*)}, \frac{\xi_h}{U^0 - U_h^R(\theta_{2,h}^*)}\right]$.

- Expression (A.12) holds with equality in the first instance. This implies that $1 \leq \pi_h \leq 0 = \pi_\ell$. Therefore, we have that $\tilde{z}_2^p(0, \pi_\ell; z_1) = 0 < z^*$, and, thus, $\lambda^p = 1$. This is an equilibrium if and only if there exist δ and λ^f such that

$$\xi_h + \delta(1-\lambda^f)\left[U_h^R(\theta_{2,h}^*) - U^0\right] = 0. \tag{IA.5}$$

That is, this is an equilibrium if and only if $\delta \geq \frac{\xi_h}{U^0 - U_h^R(\theta^*_{2,h})}$.

– Expression (A.12) holds with equality in the second instance. This implies that $\pi_h = 1 \geq \pi_\ell \geq 0$. Therefore, we have that $\tilde{z}_2^f(0, \pi_\ell; z_1) = 1 < z^*$, and, thus, $\lambda^f = 0$. This is an equilibrium if and only if there exist δ and λ^f such that

$$\xi_\ell + \delta\lambda^p \left[U_\ell^R(\theta^*_{2,\ell}) - U^0\right] = 0. \tag{IA.6}$$

That is, this is an equilibrium if and only if $\delta \geq \frac{\xi_\ell}{U^0 - U_\ell^R(\theta^*_{2,\ell})}$.

- $\lambda^p < \lambda^f$. That is, the bank originates a risky loan in the second period with strictly lower probability if it passes the stress test in the first period than if it fails the stress test in the first period. This implies that:

$$\xi_h + \delta(\lambda^p - \lambda^f) \left[U_h^R(\theta^*_{2,h}) - U^0\right] > \xi_\ell + \delta \left[U_\ell^R(\theta^*_{2,\ell}) - U^0\right], \tag{IA.7}$$

which follows because $\lambda^p - \lambda^f < 0$ and $U_h^R(\theta^*_{2,h}) - U^0 < U_\ell^R(\theta^*_{2,\ell}) - U^0 < 0$ for all $B < \underline{B} < B_\ell$. In turn, this implies that $\pi_h > \pi_\ell$. We then have $\tilde{z}_2^p(\pi_\ell, \pi_h; z_1) \leq z^* \leq \tilde{z}_2^f(\pi_\ell, \pi_h; z_1)$, implying that $\lambda^p \geq \lambda^f$, a contradiction. Therefore, no such equilibrium exists.

This concludes the proof that, in addition to those equilibria characterized in Propositions 4–7, there exist money-burning equilibria characterized above if and only if $B < \underline{B} < B_\ell$ and $\delta \geq \frac{\xi_\ell}{U^0 - U_\ell^R(\theta^*_{2,\ell})}$.

IA.2 Proposition and Proof for Reputation Updating (Section 4.2)

Proposition IA.1. *For $B > \underline{B}$, the equilibrium is one of the following three types.*

- *Reputation-building equilibrium: If the bank originates a risky loan in the first period, the type-τ regulator fails the bank at stage 2 if and only if $\theta_1 \geq \theta'_{1,\tau}$, and the bank recapitalizes if and only if it fails the stress test, where $\theta'_{1,h} > \theta'_{1,\ell}$. The bank subsequently originates a risky loan in the second period with strictly higher probability if it passes the stress test in the first period than if it fails the stress test in the first period. This equilibrium exists only if* (4) *holds (when evaluated at the equilibrium quantities). In this equilibrium, the bank originates a risky loan in the first period if and only if* (6) *holds (with* θ_τ^{FB} *replaced by* $\theta'_{1,\tau}$*).*

- *No-reputation-building equilibrium: If the bank originates a risky loan in the first period, the type-*τ *regulator fails the bank at stage 2 if and only if* $\theta_1 \geq \theta^*_{2,\tau} = \frac{\xi_\tau + C}{D}$*, and the bank recapitalizes if and only if it fails the stress test. In this equilibrium, the bank originates a risky loan in the first period if and only if* $z_1 \leq z^*$*.*

- *Always-pass equilibrium: The bank originates a risky loan in the first period; the regulator passes the bank with certainty, and the bank does not recapitalize regardless of the stress test outcome. The bank subsequently originates a risky loan in the second period if and only if* $z_1 \leq z^*$*.*

Given the type-τ regulator's stress-testing strategy $\theta_{1,\tau}$, these updated beliefs are given by:

$$z_2^{p,R}(\theta_{1,\ell}, \theta_{1,h}; z_1) = \frac{z_1 \int_0^{\theta_{1,\ell}} (1-\theta) dH(\theta)}{z_1 \int_0^{\theta_{1,\ell}} (1-\theta) dH(\theta) + (1-z_1) \int_0^{\theta_{1,h}} (1-\theta) dH(\theta)}, \tag{IA.8}$$

$$z_2^{p,0}(\theta_{1,\ell}, \theta_{1,h}; z_1) = \frac{z_1 \int_0^{\theta_{1,\ell}} \theta dH(\theta)}{z_1 \int_0^{\theta_{1,\ell}} \theta dH(\theta) + (1-z_1) \int_0^{\theta_{1,h}} \theta dH(\theta)}, \tag{IA.9}$$

$$z_2^{f,R}(\theta_{1,\ell}, \theta_{1,h}; z_1) = \frac{z_1 \int_{\theta_{1,\ell}}^{\bar{\theta}} (1-\theta) dH(\theta)}{z_1 \int_{\theta_{1,\ell}}^{\bar{\theta}} (1-\theta) dH(\theta) + (1-z_1) \int_{\theta_{1,h}}^{\bar{\theta}} (1-\theta) dH(\theta)}, \tag{IA.10}$$

$$z_2^{f,0}(\theta_{1,\ell}, \theta_{1,h}; z_1) = \frac{z_1 \int_{\theta_{1,\ell}}^{\bar{\theta}} \theta dH(\theta)}{z_1 \int_{\theta_{1,\ell}}^{\bar{\theta}} \theta dH(\theta) + (1-z_1) \int_{\theta_{1,h}}^{\bar{\theta}} \theta dH(\theta)}. \tag{IA.11}$$

In this proof, we first show that there can be two types of equilibria in which, if the bank originates a risky loan in the first period, the bank recapitalizes in the first period if and only if it fails the stress test: the no-reputation-building equilibrium and the reputation-building equilibrium. We then show that there can be an equilibrium in which the bank does not recapitalize in the first period, regardless of the stress test result. This is the always-pass equilibrium. Finally, we show that there exists no equilibrium in which the bank recapitalizes in the first period, regardless of the stress test result.

Consider, first, an equilibrium in which, if the bank originates a risky loan in the first period, the bank recapitalizes in the first period if and only if it fails the stress test.

- **No-reputation-building equilibrium**. This is an equilibrium if $\lambda^{p,R} = \lambda^{f,R}$ and $\lambda^{p,0} = \lambda^{f,0}$. This and (13) imply that the type-τ regulator passes the bank if and only if $\theta \geq \theta^*_{2,\tau}$. Since $\theta^*_{2,h} > \theta^*_{2,\ell}$, we have

$$z_2^{p,R}(\theta_{1,\ell}, \theta_{1,h}; z_1) \leq z_2^{f,R}(\theta_{1,\ell}, \theta_{1,h}; z_1), \tag{IA.12}$$

$$z_2^{p,0}(\theta_{1,\ell}, \theta_{1,h}; z_1) \leq z_2^{f,0}(\theta_{1,\ell}, \theta_{1,h}; z_1). \tag{IA.13}$$

  That is, since the high-cost regulator is softer than the low-cost regulator, it passes banks with higher risks. Therefore, given the realized payoff, passing the stress test is more indicative of the high-cost regulator (lower z_2) than failing the stress test is. A sufficient condition for such an equilibrium to exist is $z_1 \geq \overline{z}'_1$ or $z_1 \leq \underline{z}'_1$, where $\overline{z}'_1$ is defined such that $z_2^{p,R}(\theta^*_{2,\ell}, \theta^*_{2,h}; \overline{z}'_1) = z^*$ and $\underline{z}'_1$ is defined such that $z_2^{f,0}(\theta^*_{2,\ell}, \theta^*_{2,h}; \underline{z}'_1) = z^*$. If $z_1 \geq \overline{z}'_1$, then the bank invests in the safe asset in the second period, regardless of the stress test result in the first period $\lambda^{p,R} = \lambda^{f,R} = \lambda^{p,0} = \lambda^{f,0} = 0$; if $z_1 \leq \underline{z}'_1$, then the bank originates a risky loan in the second period, regardless of the stress test result in the first period $\lambda^{p,R} = \lambda^{f,R} = \lambda^{p,0} = \lambda^{f,0} = 1$.

- **Reputation-building equilibrium**. In such an equilibrium, either $\lambda^{p,R} \neq \lambda^{f,R}$ or $\lambda^{p,0} \neq \lambda^{f,0}$. This and (13) imply that $\theta'_{1,h} \geq \theta'_{1,\ell}$. Again, the properties given in (IA.12)–(IA.13) are satisfied, implying that $\lambda^{p,R} \geq \lambda^{f,R}$ and $\lambda^{p,0} \geq \lambda^{f,0}$, with at least one strict inequality. In this equilibrium, the bank's recapitalization decision must be optimal. That is, $\theta'_{1,\tau}$ must satisfy $\hat{\theta}_1^f \geq \underline{\theta}(r_1) \geq \hat{\theta}_1^p$, where $\hat{\theta}_1^f$ and $\hat{\theta}_1^p$ are defined analogously by (A.27) and (A.28) (with $\theta^*_{1,\tau}$ replaced by $\theta'_{1,\tau}$). Following similar arguments as in the proof of Proposition 4, this is equivalent to (A.29) (with $\theta^*_{1,\tau}$ replaced by $\theta'_{1,\tau}$).

Next, consider an equilibrium in which the bank does not recapitalize in the first period, regardless of the stress test result. Following similar arguments as in the proof of Proposition 4, we can show that there is an equilibrium in which the regulator passes the bank with certainty. This is the **always-pass equilibrium**. Moreover, following similar arguments as in the proof of Proposition 4 and the proof of Proposition 6, respectively, this equilibrium is not a PSE whenever a reputation-building equilibrium exists or whenever a no-reputation-building equilibrium exists. Furthermore, following arguments similar to those in the proof of Lemma 2, Assumption 7 implies that there exists no equilibrium in which the bank does not recapitalize in the first period, regardless of the stress test result, other than the always-pass equilibrium (e.g., a money-burning equilibrium).

Finally, following similar arguments as in the proofs of Propositions 4 and 6, no equilibrium exists in which the bank faces a withdrawal threat, regardless of the stress test result (but is only able to recapitalize if it fails the stress test), because it is not a PSE.

IA.3 Propositions and Proofs for Extension with Bank Runs (Section 4.3)

In this extension, we modify Assumptions 4, 5 and 6 as follows:

Assumption 4'. $\hat{\xi}_h < (1-\bar{\theta})R - L$.

Analogous to Assumption 4, this assumption assumes that the effective cost of failing the bank $\hat{\xi}_\tau$, which takes into account the fact that the lending facility only becomes available with probability γ, is less than the gain from not liquidating the bank.

Assumption 5'. $1-\bar{\theta} < 1-\hat{\theta}^*_\tau < (1-\mathbb{E}[\theta])L$ *for all* $\tau \in \{h, \ell\}$*, where* $\hat{\theta}^*_\tau$ *is defined in* (15)*.*

Analogous to Assumption 5, this assumption ensures that in the equilibrium in the second period described in Proposition IA.2, the bank passes and fails the stress test with strictly positive probabilities.

Assumption 6'. $\int_{\hat{\theta}^*_h}^{\bar{\theta}} [\gamma C + (1-\gamma)(\theta D - L)]\, dH(\theta) < (1-\mathbb{E}[\theta])R - R_0 < \int_{\hat{\theta}^*_\ell}^{\bar{\theta}} [\gamma C + (1-\gamma)(\theta D - L)]\, dH(\theta)$, *where* $\hat{\theta}^*_\tau$ *is defined in* (15)*.*

Analogous to Assumption 6, this assumption allows us to focus on the interesting parameter space in which the bank's investment decision in the second period is sensitive to the regulator's reputation in equilibrium, as in the baseline model.

The following proposition characterizes the one-period benchmark. As in the baseline model, this coincides with the equilibrium outcome in the second period of the two-period game.

Proposition IA.2. *If the regulator's lending facility is only available with probability* γ*, there exists a unique equilibrium in the one-period benchmark as follows:*

- *If the bank originates a risky loan at stage 1, the type-*τ *regulator fails the bank at stage 2 if and only if* $\theta \geq \hat{\theta}^*_\tau$ *(defined in* (15)*), where* $\hat{\theta}^*_\tau > \hat{\theta}^{FB}_\tau \equiv \frac{\hat{\xi}_\tau + C}{D}$*, and the bank recapitalizes if and only if it fails the stress test and a lending facility is available.*

- *The bank originates a risky loan at stage 1 if and only if*

$$
\begin{aligned}
(1-\mathbb{E}[\theta])R - R_0 \quad \geq \quad & z\int_{\hat{\theta}_\ell^*}^{\bar{\theta}} [\gamma C + (1-\gamma)(\theta D - L)]\, dH(\theta) \\
& +(1-z)\int_{\hat{\theta}_h^*}^{\bar{\theta}} [\gamma C + (1-\gamma)(\theta D - L)]\, dH(\theta). \qquad \text{(IA.14)}
\end{aligned}
$$

First, the regulator's stress-testing strategy $\hat{\theta}_\tau^*$ follows directly from the discussion in Section 4.3.

Second, the bank's investment decision at stage 1 is derived taking into account the risk of a run following failing the stress test. There are two opposing effects. On the one hand, the bank is less likely to originate a risky loan because of the possibility of a run when failing to recapitalize. On the other hand, the bank is more likely to originate a risky loan because it anticipates a more lenient stress test.

Finally, under Assumptions 4' and 5', the existence and uniqueness of this equilibrium follows the same logic as in the proof of Proposition 2.

We now turn to characterizing the equilibrium in the first period of the two-period model.

Proposition IA.3. *There exists $\hat{\underline{B}}$, such that for $B > \hat{\underline{B}}$, the equilibrium is one of the following three types.*

- *Reputation-building equilibrium: If the bank originates a risky loan in the first period, the type-τ regulator fails the bank at stage 2 if and only if $\theta_1 \geq \hat{\theta}_{1,\tau}$,where $\hat{\theta}_{1,\tau} \neq \hat{\theta}_\tau^*$. The bank recapitalizes if and only if it fails the stress test and a lending facility is available, where $\hat{\theta}_{1,h} > \hat{\theta}_{1,\ell}$. The bank subsequently originates a risky loan in the second period with strictly higher probability if it passes the stress test in the first period than if it fails the stress test in the first period. This equilibrium exists only if* (4) *holds (when evaluated at the equilibrium quantities). In this equilibrium, the bank originates a risky loan in the first period if and only if* (IA.14) *holds (with $\hat{\theta}_\tau^*$ replaced by $\hat{\theta}_{1,\tau}^*$ and z replaced by z_1).*

- *No-reputation-building equilibrium: If the bank originates a risky loan in the first period, the type-τ regulator fails the bank at stage 2 if and only if $\theta_1 \geq \hat{\theta}_\tau^*$, and the bank recapitalizes if and only if it fails the stress test and a lending facility is available. The bank subsequently* *originates a risky loan in the second period if and only if* (IA.14) *holds (with* z *replaced by the updated beliefs about the regulator's type). In this equilibrium, the bank originates a risky loan in the first period if and only if* (IA.14) *holds.*

- *Always-pass equilibrium: The bank originates a risky loan in the first period; the regulator passes the bank with certainty, and the bank does not recapitalize regardless of the stress test outcome. The bank subsequently originates a risky loan in the second period if and only if* $z_1 \leq z^*$.

Under the modified assumptions, we can follow the same arguments as in the baseline model, and show that all results in Corollary 1, Lemma 3, and Propositions 4–7 continue to hold for this extension of the model.